Credit: NASA (Bill Hrybyk)

# Focusing in on the High Energy Universe

## II: Introduction to X-ray Optics


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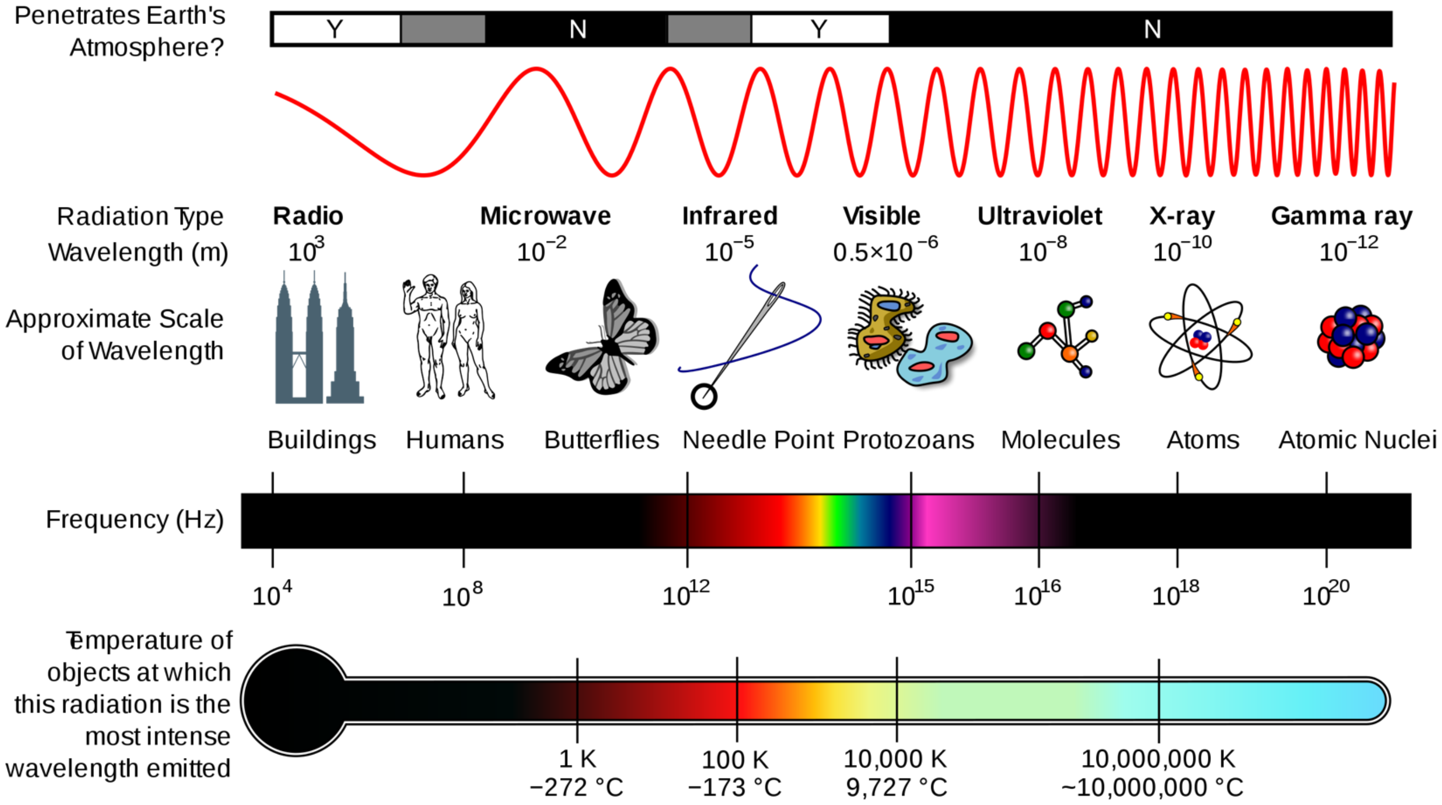
A 100 $mm^2$ Wolter-I mirror segment

Universities Space Research Association

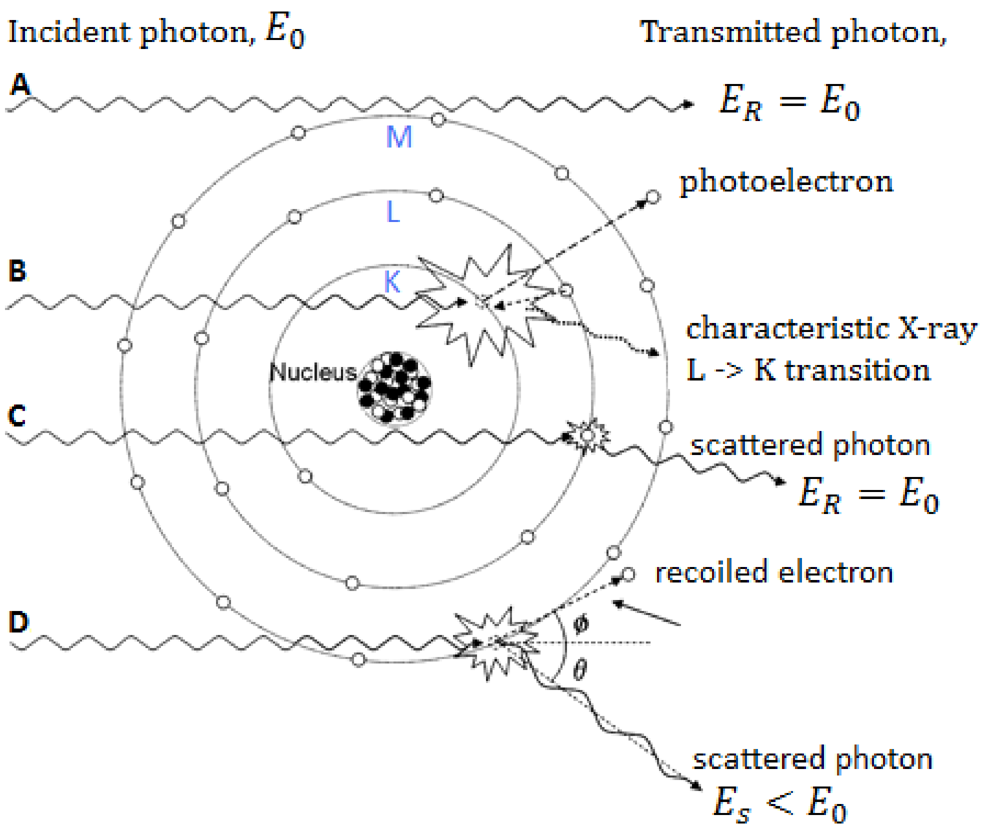
# Light – Electromagnetic (EM) Radiation




Credit: NASA / Wiki CC Inductiveload

- **Wave-particle duality**
  - **Propagates as waves**
  - **Energy imparted by waves absorbed like particles at single sites: *photon* (quanta of light)**
- **Optics is the study of light.**

# X-rays and Atoms




**A. TRANSMITTED UNAFFECTED**
No interaction

**B. PHOTOELECTRIC ABSORPTION**
Collision with a tightly bound inner-shell electron

**C. RAYLEIGH SCATTERING**
Elastic collision with a bound outer-shell electron

**D. COMPTON SCATTERING**
Inelastic collision with weakly bound outer-shell electron

*Image Credit: O'Hea, I., MSc Thesis, Open University (UK), 2015*

# Optics

- "Optics" (from the Greek word for appearance - "οπτική")
- Lenses made from quartz as early as 2000 BC (i.e. Nimrud lens from 700 BC Assyria).
- Branch of physics that focuses on the behavior and properties of light (*incl*. manipulation, detection, etc).
- Two branches of (Classical) Optics:
  - Geometrical (Ray) Optics
    - Light propagates as straight lines
  - Physical (Wave) Optics
    - Light propagates as an electromagnetic wave

The main mirror of the HST (Credit: NASA)

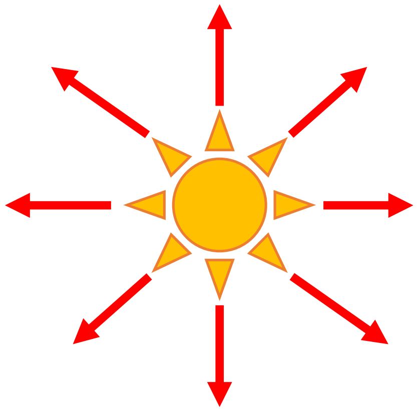
# Geometrical (Ray/Particle) Optics

*Credit: NASA/ Goddard/ Anne Koslosky*

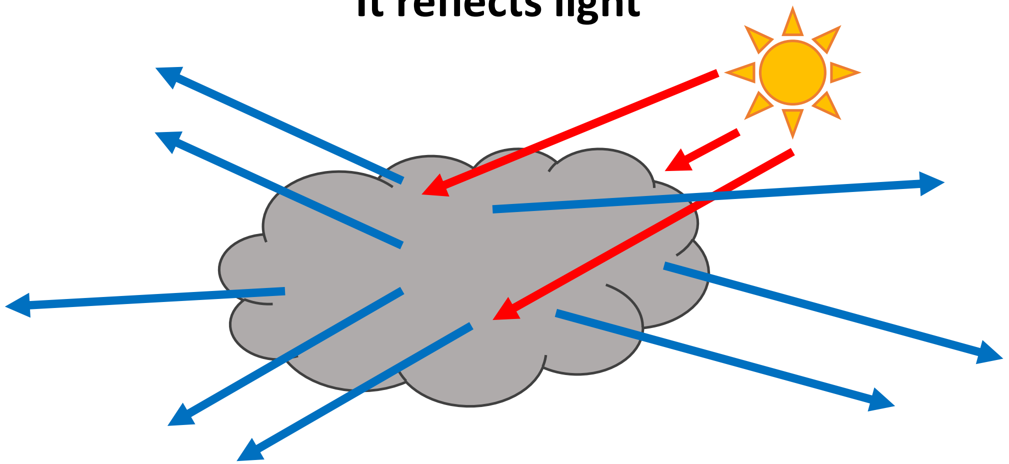
# Why do we see things?

- Objects are a source of light rays that originate from everywhere on the object with each point sending rays omnidirectionally.
- Some wavelengths are reflected, others absorbed giving colour which can be detected with correct sensor.
- Objects can be self-luminous (i.e. Sun, flames, lightbulb) or reflective. Most objects are reflective.

**Self-luminous object (active)**
**It emits light**



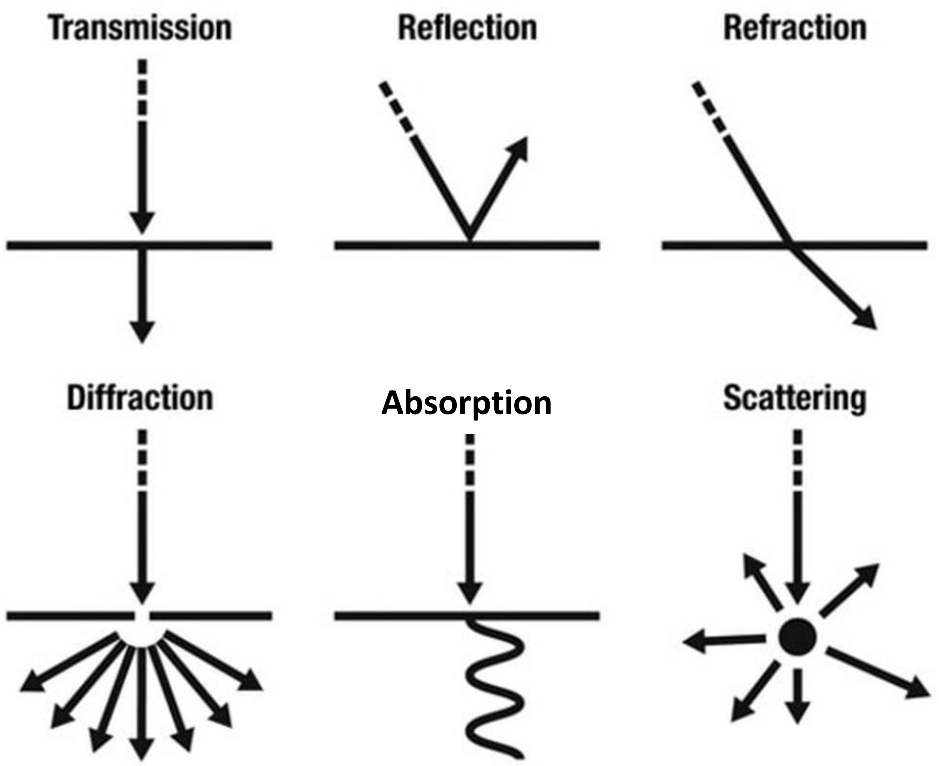
**Self-luminous object (passive)**
**It reflects light**



# The “Ray” Model of Light

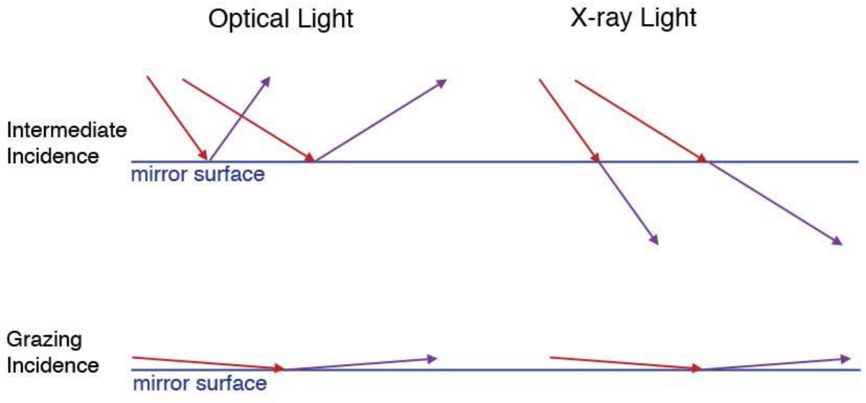
- Light interacts with matter in different ways:
  - <u>Between</u> two materials, it can be *reflected* or *refracted*
  - <u>Within</u> a material, light can be *scattered* or *absorbed*
  - Or, it can be *transmitted* or *diffracted* through the medium.

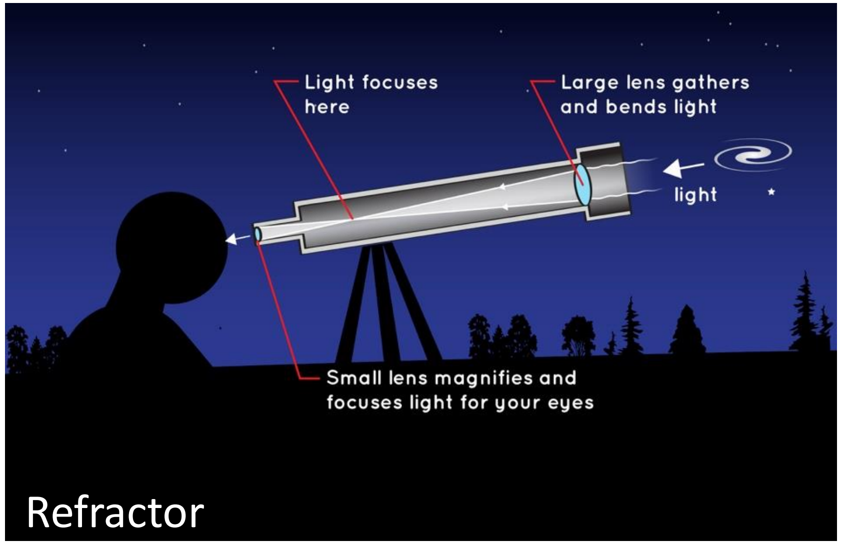
- How incident light interacts with matter depends on how transparent the material/medium is to the wavelength of light and the surrounding material.

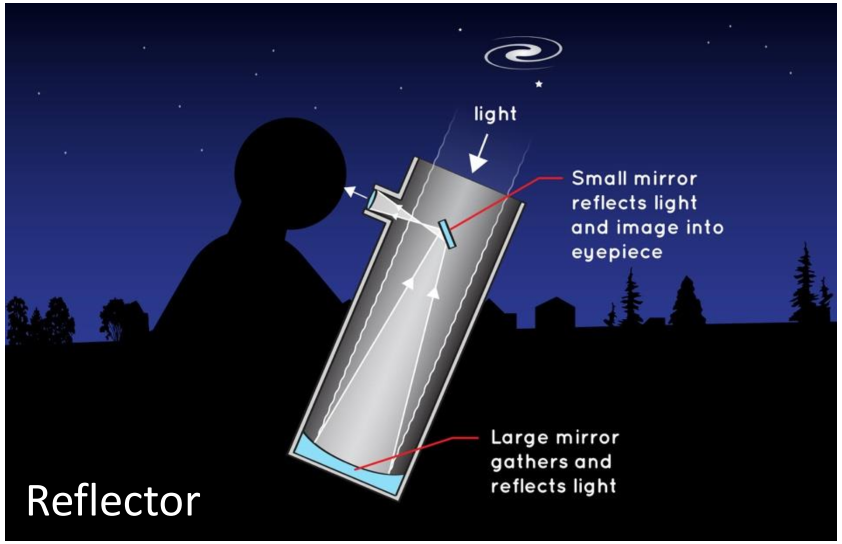


# Mirrors and Telescopes

- Conventional telescopes use **reflection** or **refraction** at normal/intermediate incidence (i.e. visible light).
- *Not* applicable for X-rays as rays travelling perpendicular to mirror are absorbed or transmitted (no reflection).
- X-ray telescope work use grazing incidence optics (mirror reflection at shallow angles) for total external reflection.
- Scattering of any wave by electrons is coherent only if the ***angle of incidence equals the angle of reflection***; $\boldsymbol{\phi_i = \phi_o}$

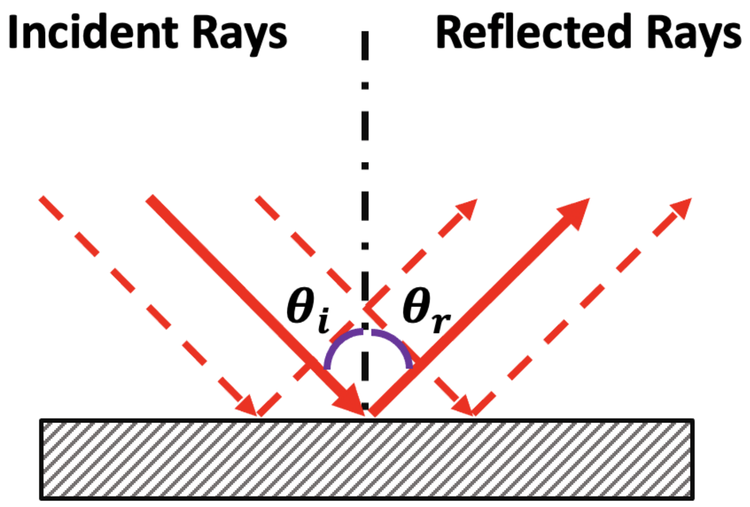



*All Image Credit: NASA*

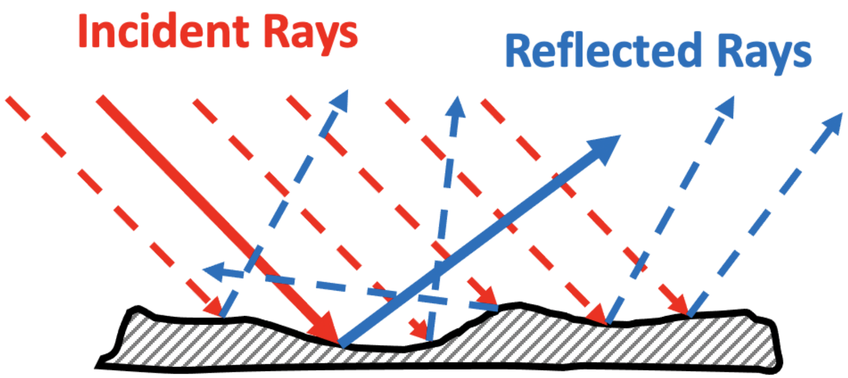
## Specular Reflection
(Smooth Surface)




$$\theta_i = \theta_r$$

- **Law of reflection:**
  - Incident & reflected about a normal line, rays are in the same plane
  - Angle of incidence ($\theta_i$) = angle of reflection ($\theta_r$)

## Diffuse Reflection
(Rough Surface)

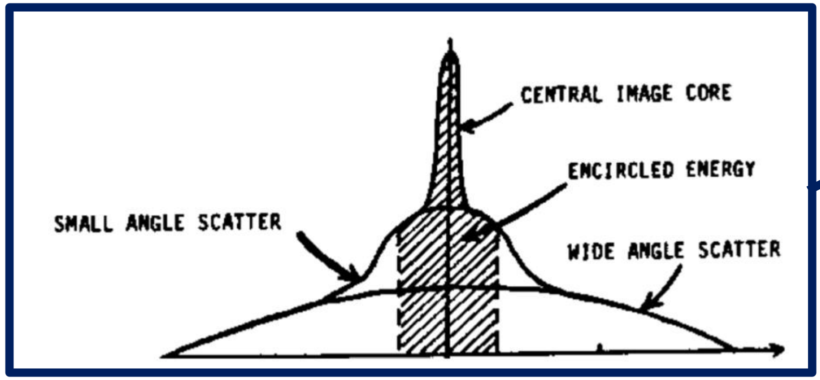



For a rough surface, the irregularities on the surface at each point cause light to be reflected randomly in many different directions

Examples: The wall, hands, etc.

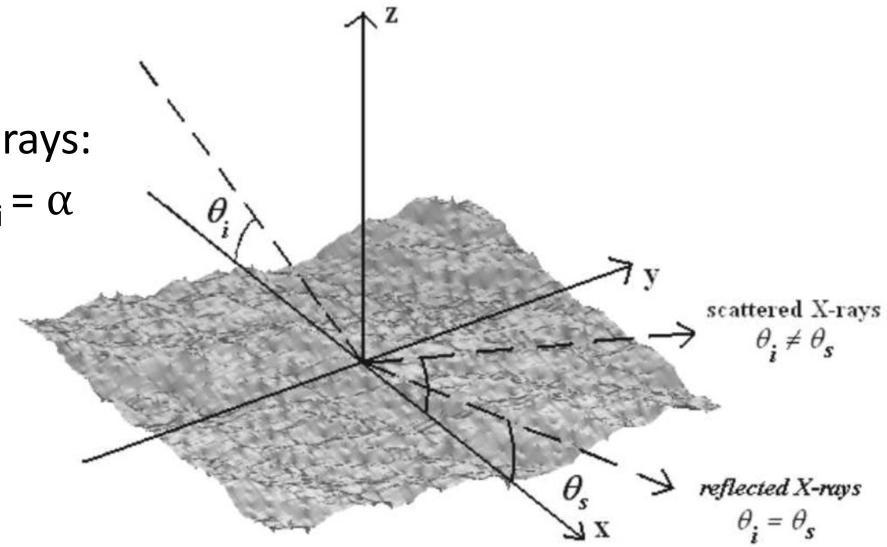
# Reflection - *Scattering*

- Surfaces are not infinitely smooth, leads to **scattering**.
- Predominately in the plane of the incident X-ray and the normal to the surface, dominates by factor of $1/sin\alpha$.
- Scattering is asymmetric. Backward scattering no more than -α (transmission). Forward scattering unlimited.
- X-rays scattered on the atomic level (i.e. electrons) result in poor image quality.
- Can't be treated *exactly* as complex. Must consider a statistical description.

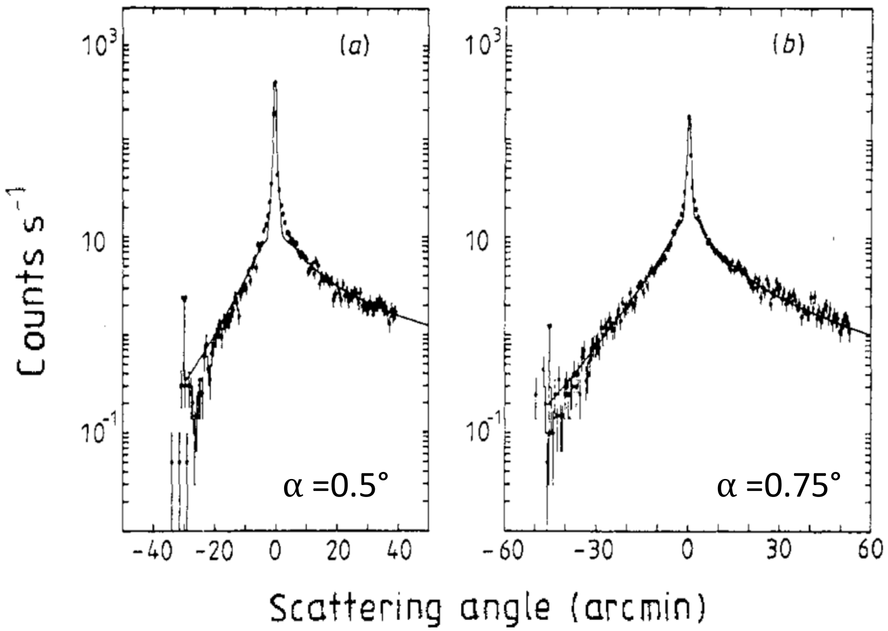



D.Spiga A&A, 7228, 2007 https://arxiv.org/pdf/0704.1612.pdf

X-rays:
$\theta_i = \alpha$




Scattering Distributions at 9.3 keV on "Kanigen" (90% Ni, 10% P)




*Adapted from: Lenzen,R.,1978 PhD Thesis Universitat Munchen*

# Reflection - *Scattering*

- Scattering theory assumes a Gaussian distribution of random irregularities in surface heights $h(x)$ and with spatial frequency $f$, characterized by a power spectral density function:

$$2W_1(f) = \left| \int e^{i2\pi x f} h(x)dx \right|^2$$

- Relative total scattered intensity:

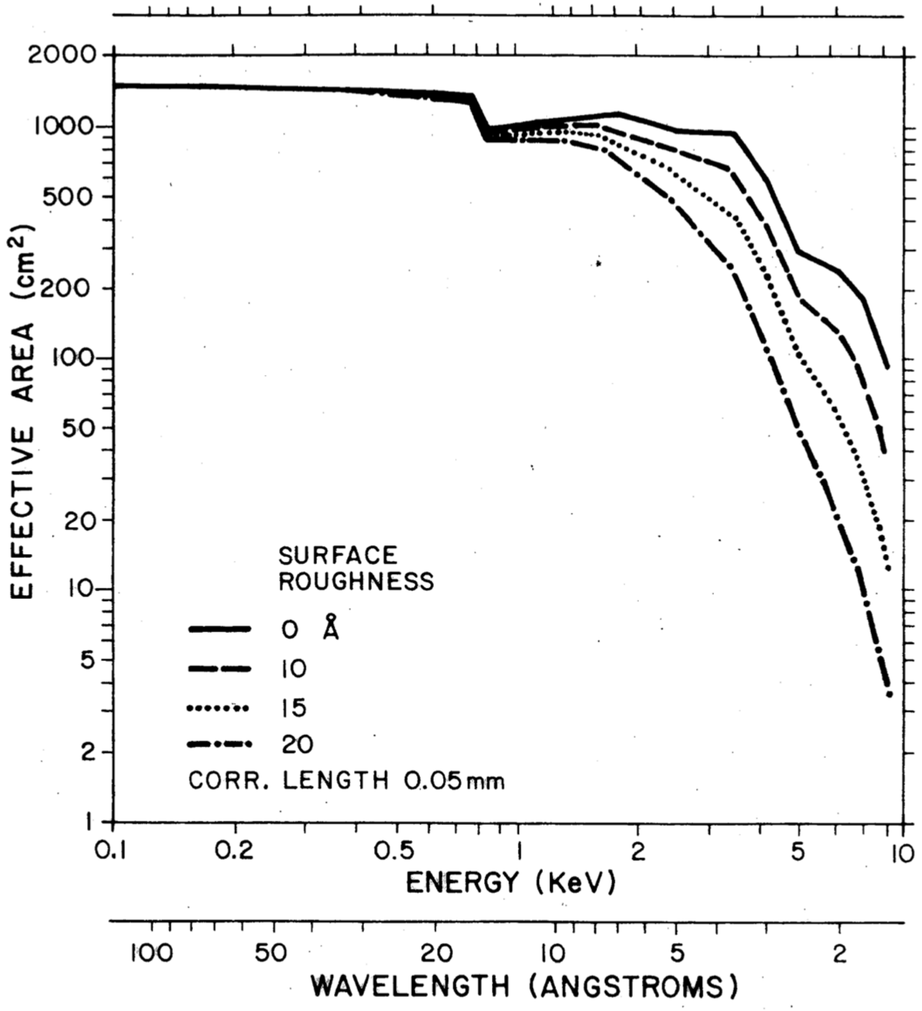
$$1 - e^{-(4\pi\sigma sin\alpha/\lambda)^2} \sim 4\pi\sigma sin\alpha/\lambda^2$$

when the exponent is small. The scattered intensity (relative to the total power in the focal plane:

$$\psi(\epsilon) = 2W_1(f)8\pi(\sin\alpha)^4/f\lambda^4$$

- The RMS roughness of the surface (σ) can be defined as:

$$\sigma^2 = \int 2W_1(f)df$$

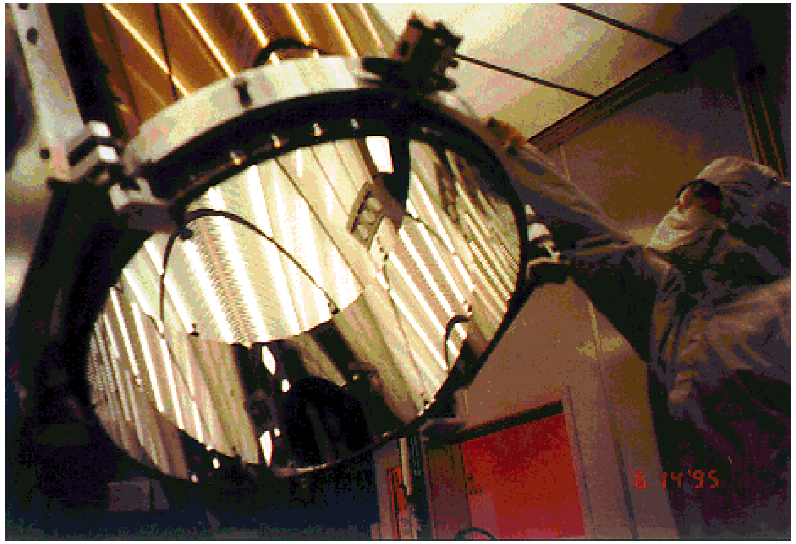



**More info:** Aschenbach, B. 1985, Rep. Prog. Phys. 48, 579; https://doi.org/10.1088/0034-4885/48/5/001

# Reflection - *Scattering*

- No perfect interface from a vacuum to infinitely thick reflecting layer
  - i.e. Chandra mirrors. Cr to hold Ir reflecting layer to Zerodur glass mirror substrate.
  - **Unwanted but inadvertent overcoat of contamination!**

- More on telescopes in the next presentation…



*Credit: CXC/SAO/NASA*

# Refraction

- Snell's Law of Refraction:

$$\frac{sin\phi_i}{sin\phi_r} = \frac{n_2}{n_1} = \frac{v_2}{v_1}$$

Where $n_i = \frac{c}{v_i}$ $and < 1\ (X - rays)$.

- Refractive index is complex for soft X-rays due to appreciable absorption by atoms:

$$n = 1 - \delta + i\beta,$$

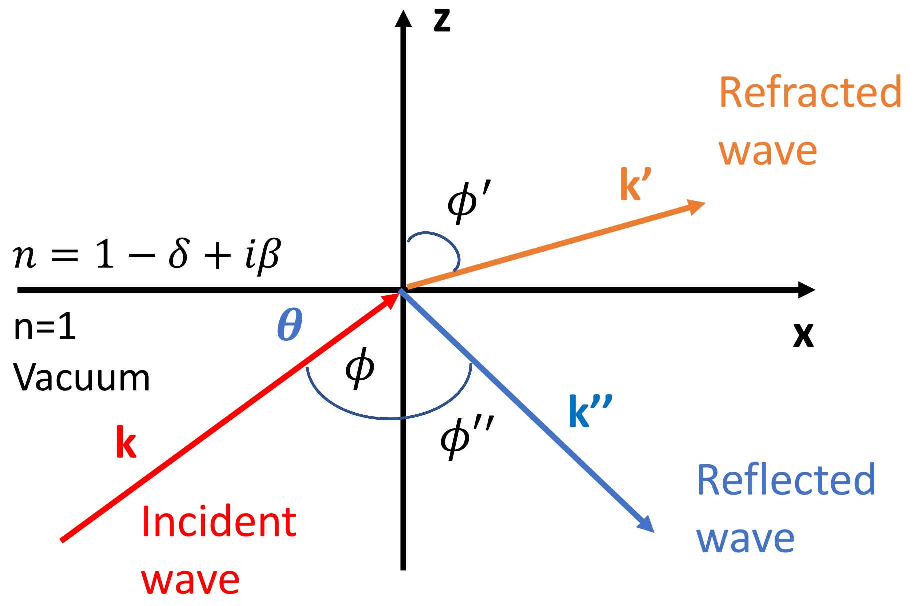
where $\delta$ is *decrement. For X-rays this is positive and small* ($10^{-4\text{-}6}$).

- In the direction of propagation:

$$E(r,t) = E_0 e^{-i[\omega t - \left(\frac{\omega}{c}\right)(1-\delta+i\beta)r]} = E_0 e^{-i\omega\left(t-\frac{r}{c}\right)} e^{-i\left(\frac{\omega\delta}{c}\right)r} e^{-\left(\frac{\omega\beta}{c}\right)r}$$

- We have a wave propagating in vacuum, a phase shift caused by the small real deviation from 1, and attenuation caused by the imaginary part of the refractive index (wave amplitude decay).

**$\theta$ is the grazing angle, < $\theta_c$**




Incident Wave: $E = E_0 e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$
Reflected Wave: $E' = E_0' e^{-i(\omega t - \boldsymbol{k'}\cdot\boldsymbol{r})}$
Refracted Wave: $E'' = E_0'' e^{-i(\omega t - \boldsymbol{k''}\cdot\boldsymbol{r})}$

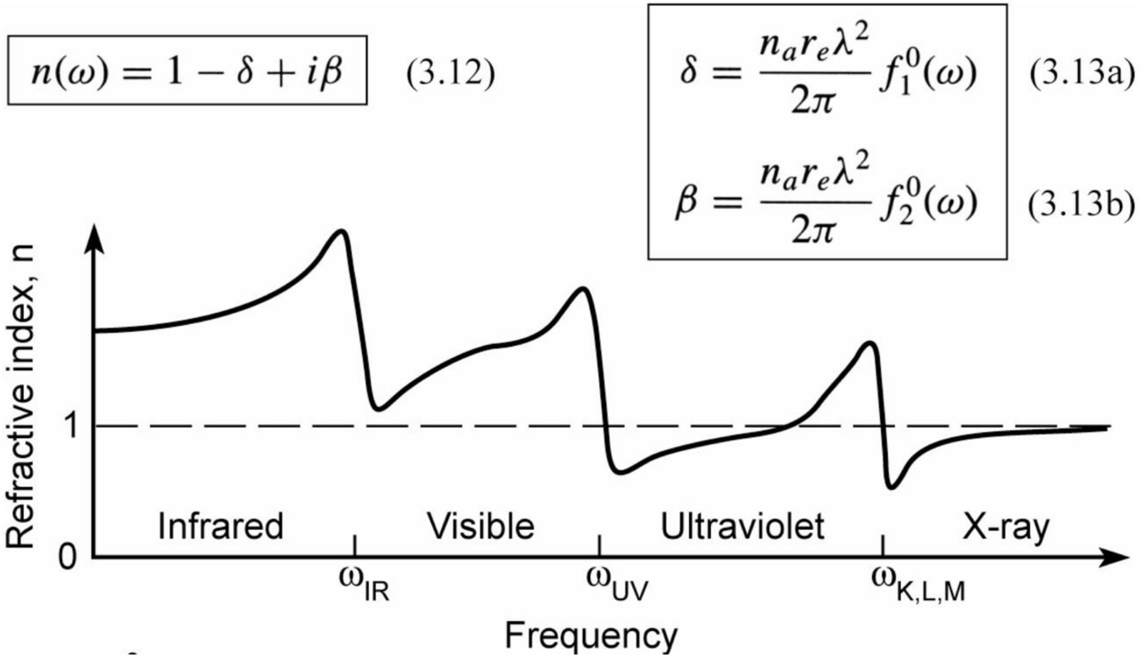
All the waves have same frequency (ω) and |**k**| as:

$$|\mathbf{k}| = |\mathbf{k}''| = \frac{\omega}{c}$$

Refracted wave has phase velocity:

$$V_\phi = \frac{\omega'}{k'} = \frac{c}{n}, \qquad thus\ |\mathbf{k}| = |\mathbf{k}''| = \frac{\omega}{c}(1 - \delta + i\beta)$$

# Refraction




Where, $n_a$ is Avogadro's number and $r_e$ is the classical electron radius, defined as:

$$r_e = \frac{e^2}{4\pi\epsilon_0 mc^2}$$

- The complex scattering factor $f_i^0(\omega) = f_1^0 - if_2^0$ is a sum of all the electrons weighted inversely, by the difference of the squares of the photon frequency and the resonant frequency of the electron, considering a damping factor. It can be calculated from the atomic photon absorption cross-section, $\mu_a$:

$$f_1^0 = Z + \frac{1}{\pi r_e hc}\int_0^\infty \frac{x^2 \mu_a(x)}{E^2 - x^2} dx + \Delta f_{rel} \qquad f_2^0 = \frac{E\mu_a(E)}{2\pi r_e hc}$$

Kramer-Kronig Relations

where $\Delta f_{rel}$ is a relativistic correction term which is negligible in the soft X-ray range


D. Attwood, UC Berkeley (2009) https://people.eecs.berkeley.edu/~attwood/sxr2009/lecnotes/04_Reflection_And_Refraction_2009.pdf
Aschenbach, B. 1985, Rep. Prog. Phys. 48, 579; https://doi.org/10.1088/0034-4885/48/5/001

# Critical Grazing Angle

- The largest possible angle of incidence resulting in a refracted ray is the **critical angle ($\boldsymbol{\phi_c}$)**.
- Using Snell's law of $n \doteq 1 - \delta$ and assuming $\beta \longrightarrow 0$:

$$sin\phi' = \frac{sin\phi_c}{1-\delta}$$

- Using the limit $\phi' \longrightarrow \frac{\pi}{2}$, total external reflection:

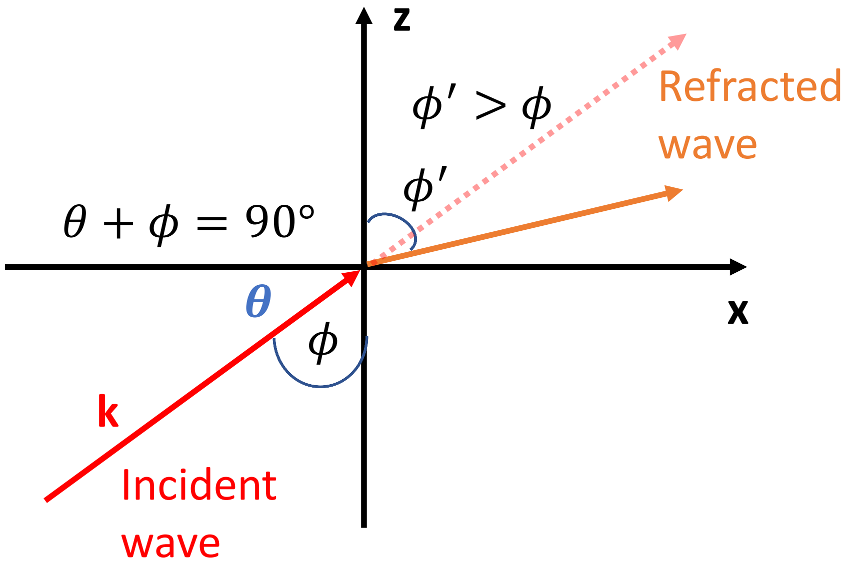
$$\theta_c = \sqrt{2\delta} = \sqrt{\frac{n_a r_e \lambda^2 f_1^0(\lambda)}{\pi}}$$

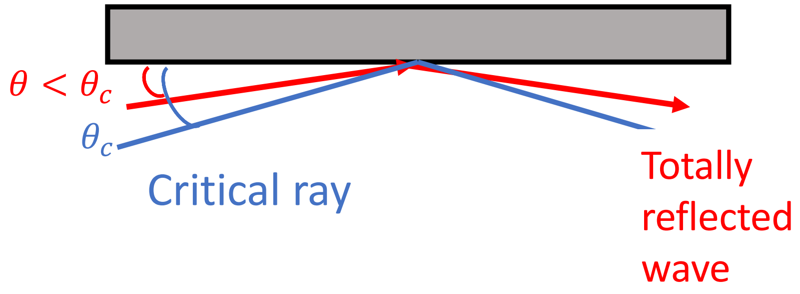
atomic density varies slowly among natural elements, to 1st order:

$$\theta_c \propto \lambda\sqrt{Z} \; or \; \frac{\sqrt{Z}}{E}$$

$f_1^0$ is approximated by Z and is a complicated function of wavelength or photon energy

- $\boldsymbol{\phi_c}$ decreases with increasing energy
- Higher Z materials reflect higher energies for fixed grazing angles
- Higher Z material have a larger critical angle at any energy




Glancing incidence ($\theta < \theta_c$) and total external reflection

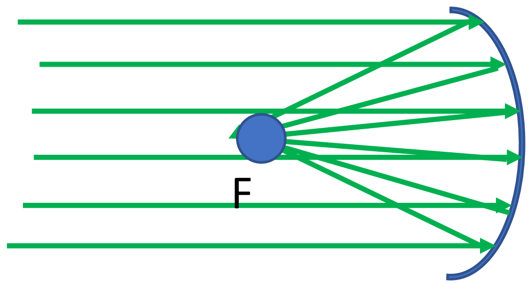
# Lenses

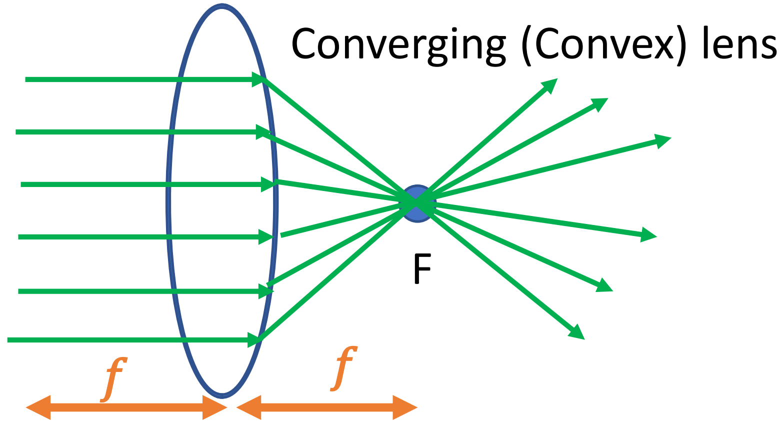
- Wolter-I telescope like a positive convex lens.
- Normal incidence optical telescopes concave mirrors.
- Lens Maker's Equation:

$$\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$$

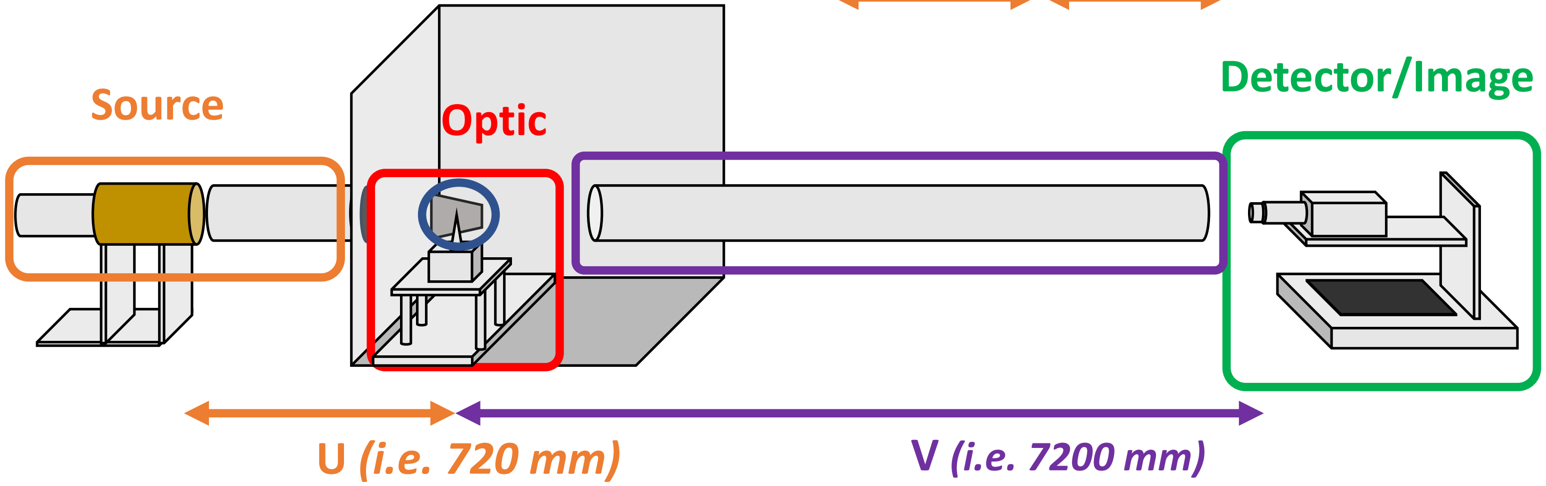
- Magnification is just the ratio of the image distance to object distance.

Concave "Converging" mirror

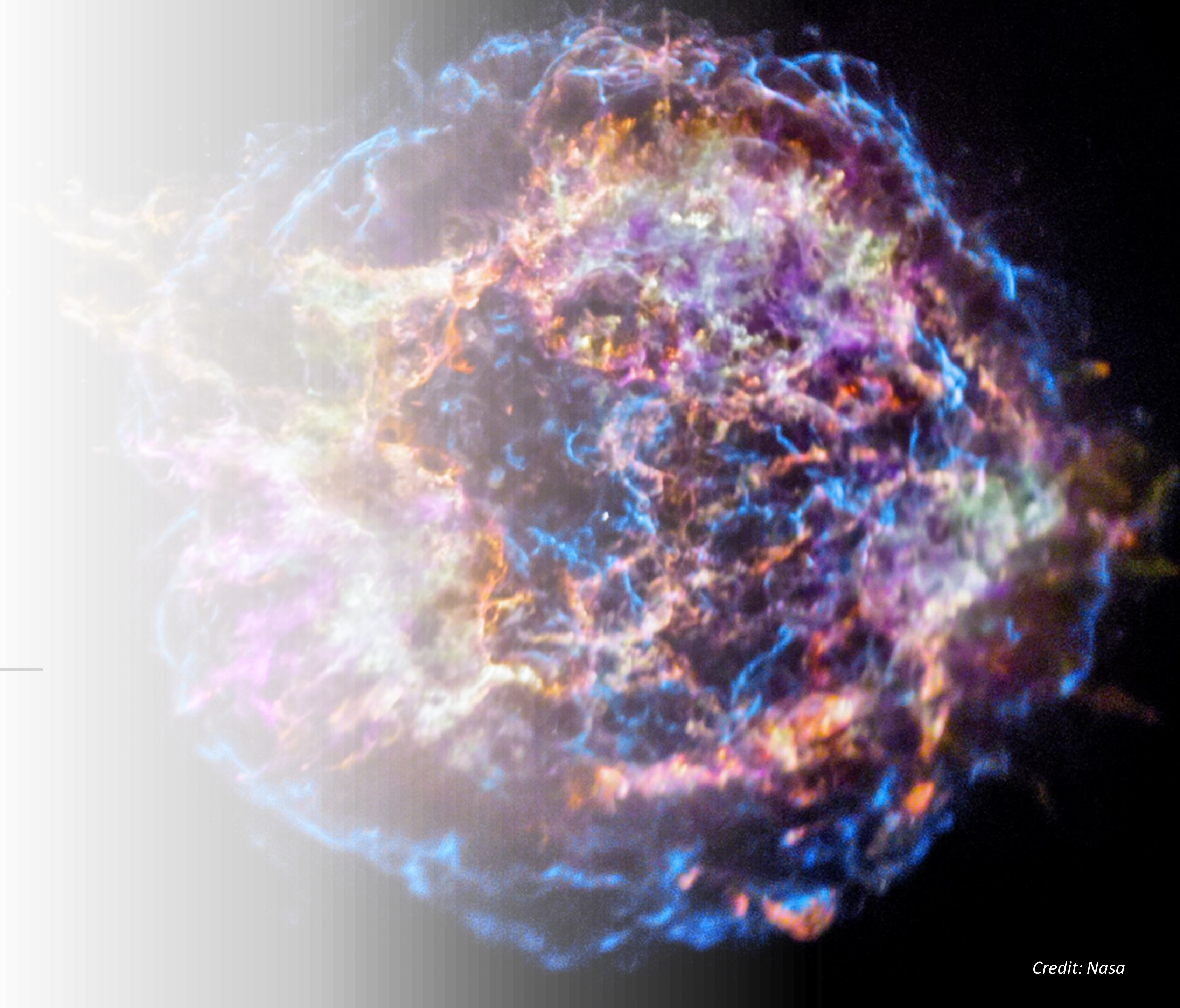



**Magnification** =**V**/**U**
*i.e. 7200/720=10x*



# Physical (Wave) Optics



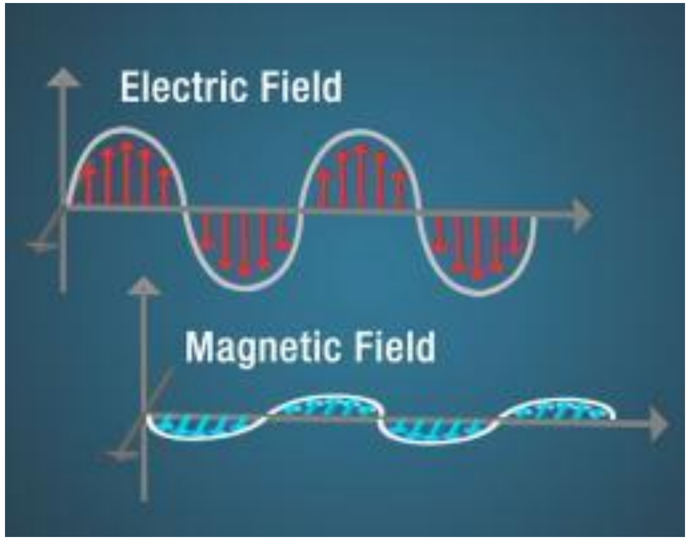
*Credit: Nasa*

# Physical (Wave) Optics

- Particle theory of light doesn't work when understanding phenomena like **interference**, **diffraction** and **polarization**. <u>We need to treat light like a wave</u>.

- Light as an electromagnetic wave with oscillating electrics (**E**) and magnetic (**B**) fields perpendicular to each other propagating through space with equal amounts of stored energy.

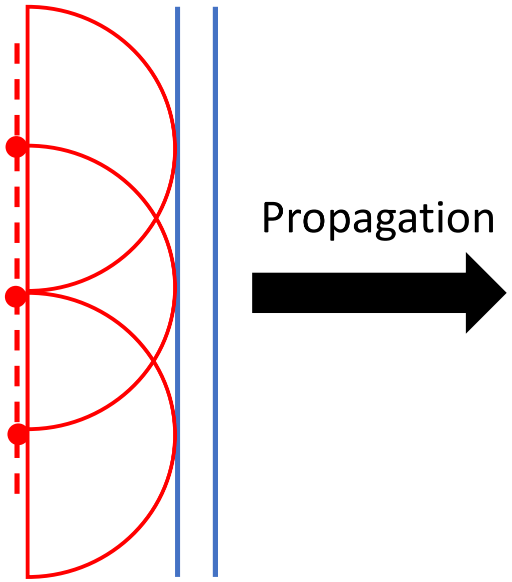
*Credit: NASA*

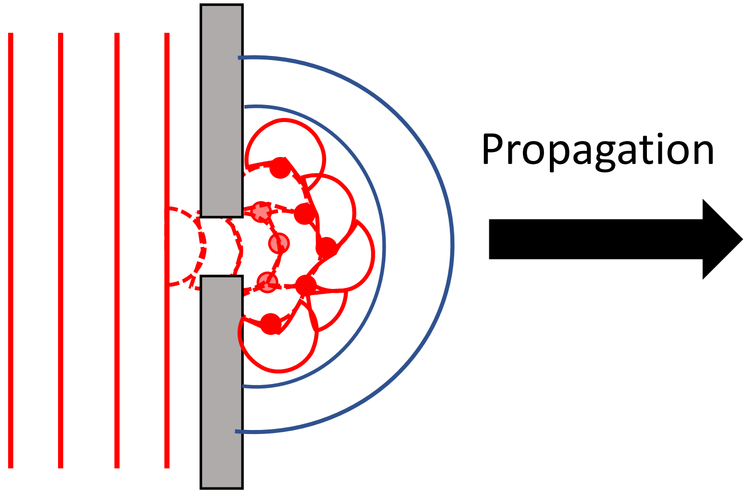



For interactions with matter, the electric component does most the work

# Wave Optics: Huygen’s Principle

- *“Any point along a wavefront can act as a new source of spherical waves, each of which has the same frequency and amplitude as its parent.”*
- For a wavefront moving with $v\Delta t$ and no obstacle, the wavefront can appear linear. (tangent to all wavelets)
- When introducing a gap in a wall, the incoming wavefront forms a circular wave front around the center of the slit.



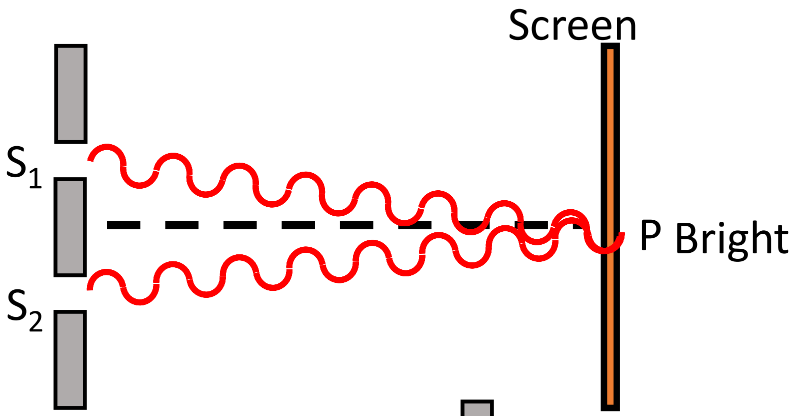
# Wave Optics: Interference

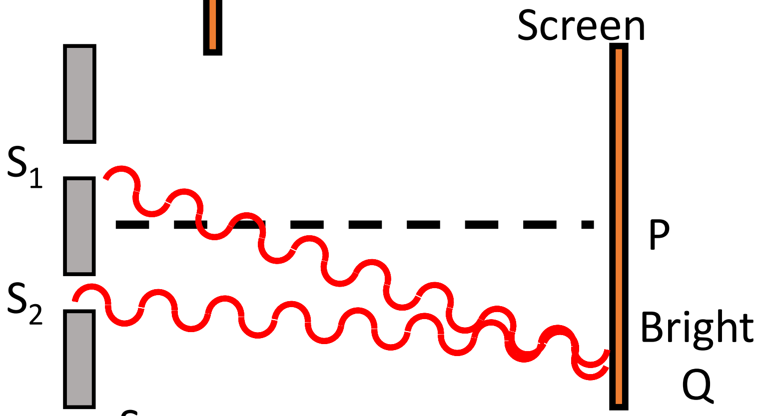
- Interference of 2 light waves:
  - must be **in phase** with each other (coherent)
  - **identical** wavelengths

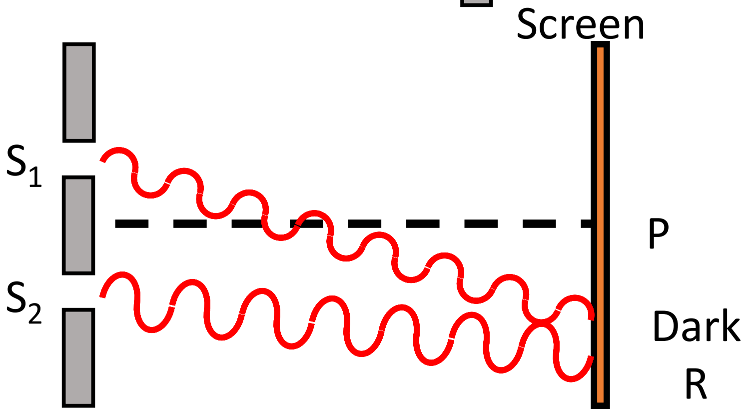
- Waves through $S_1$ and $S_2$ spread out and **interfere** with each other, producing bright and dark fringes.

- **Constructive** Interference:
  - Superposition of waves (from $S_1$ and $S_2$) in phase – bright fringe

- **Destructive** Interference:
  - Superposition of waves (from $S_1$ and $S_2$) out of phase (180°) – dark fringe



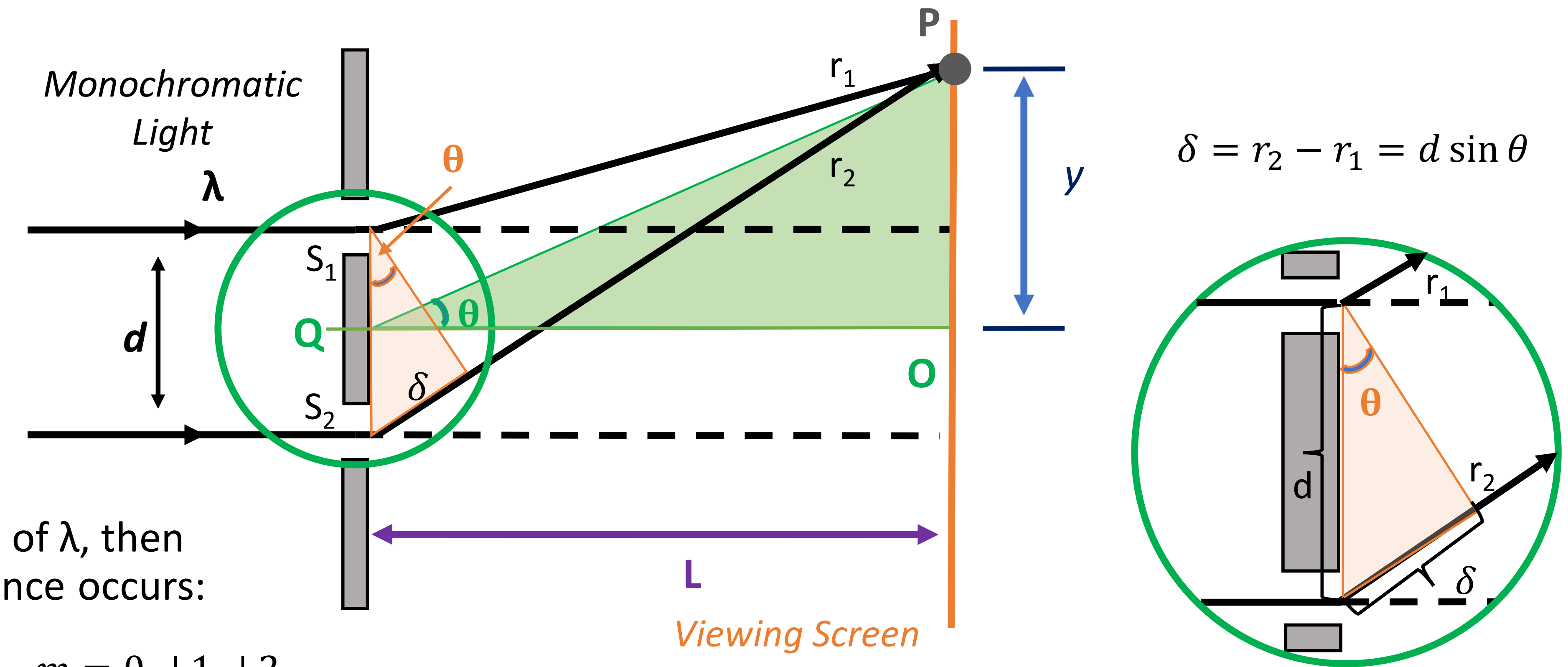



# Wave Optics: Constructive Interference

- Light from Slit 2 travels further than $S_1$. Path length difference is $d \sin\theta$ .

$$y = L\tan\theta > L\sin\theta$$

$$y_{bright} = \left(\frac{\lambda L}{d}\right) \boldsymbol{m}$$




$$\delta = r_2 - r_1 = d\sin\theta$$

- If $d\sin\theta$ is a multiple of λ, then constructive interference occurs:

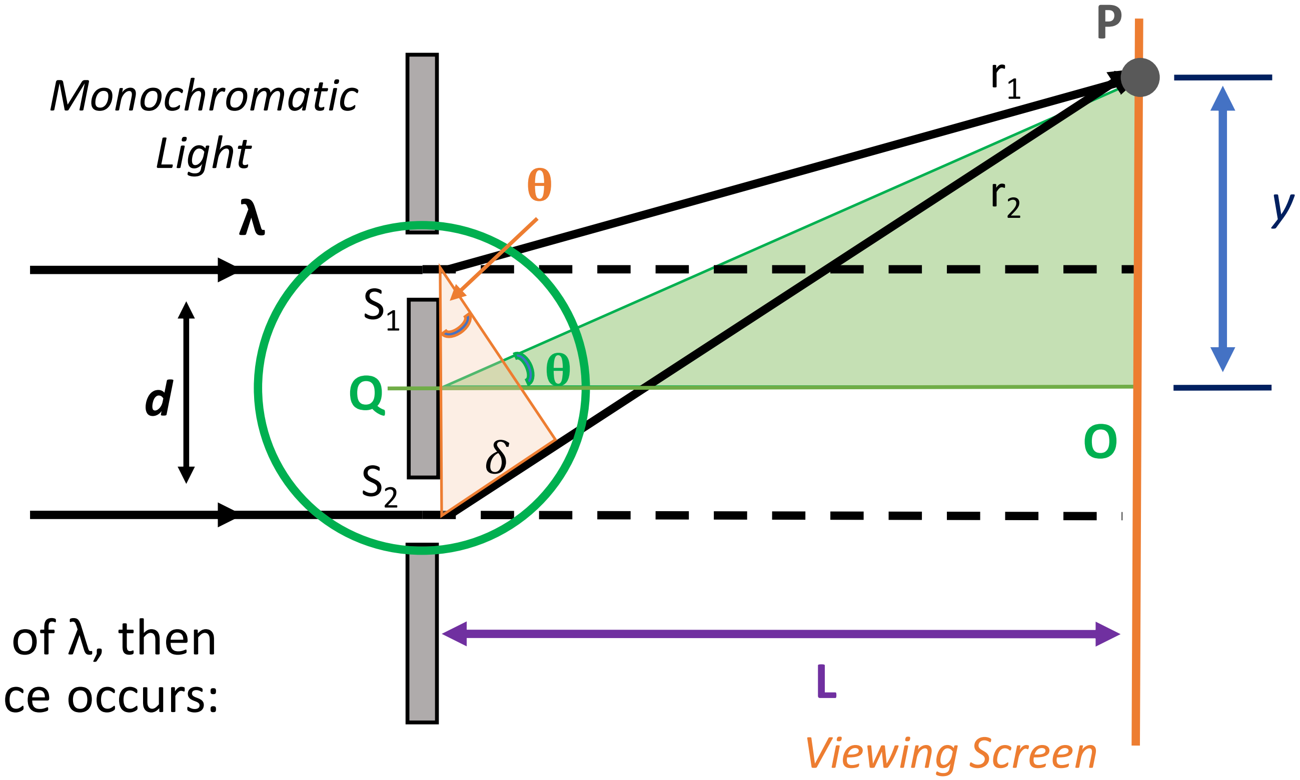
$$d\sin\theta = \boldsymbol{m}\lambda \qquad m = 0, \pm1, \pm2, \ldots$$

# Wave Optics: Destructive Interference

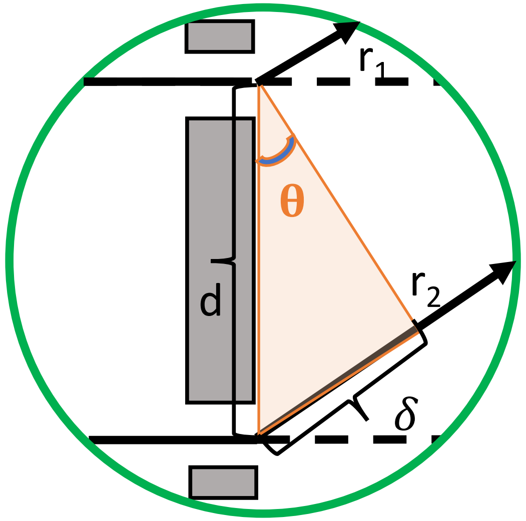
- Light from Slit 2 travels further than $S_1$.
  Path length difference is $d \sin\theta$ .

$$y = L \tan\theta > L \sin\theta$$

$$y_{dark} = \left(\frac{\lambda L}{d}\right)(m + \frac{1}{2})$$




$$\delta = r_2 - r_1 = d \sin\theta$$




- If $d \sin\theta$ is a multiple of λ, then destructive interference occurs:

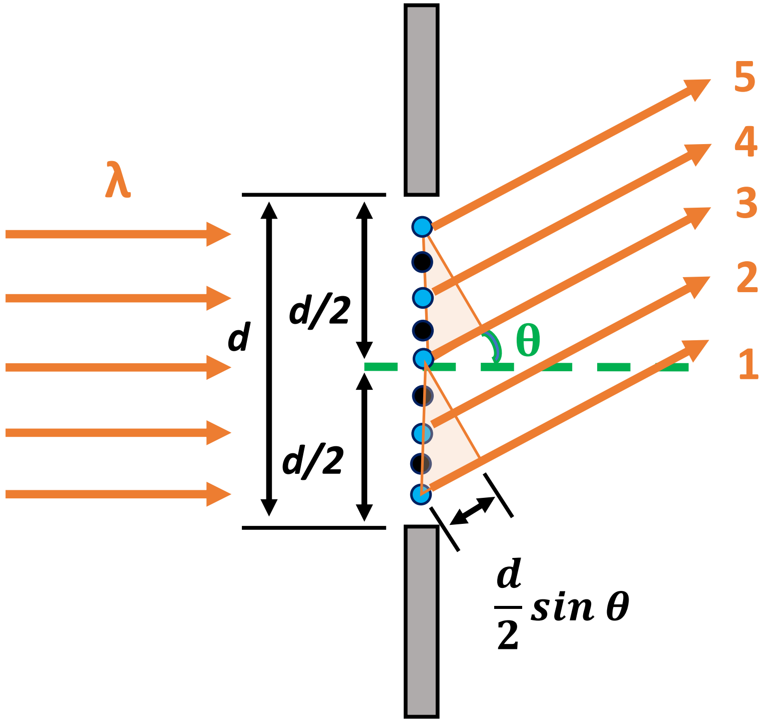
$$d \sin\theta = (m + \frac{1}{2})\lambda \qquad m = 0, \pm 1, \pm 2, \ldots$$

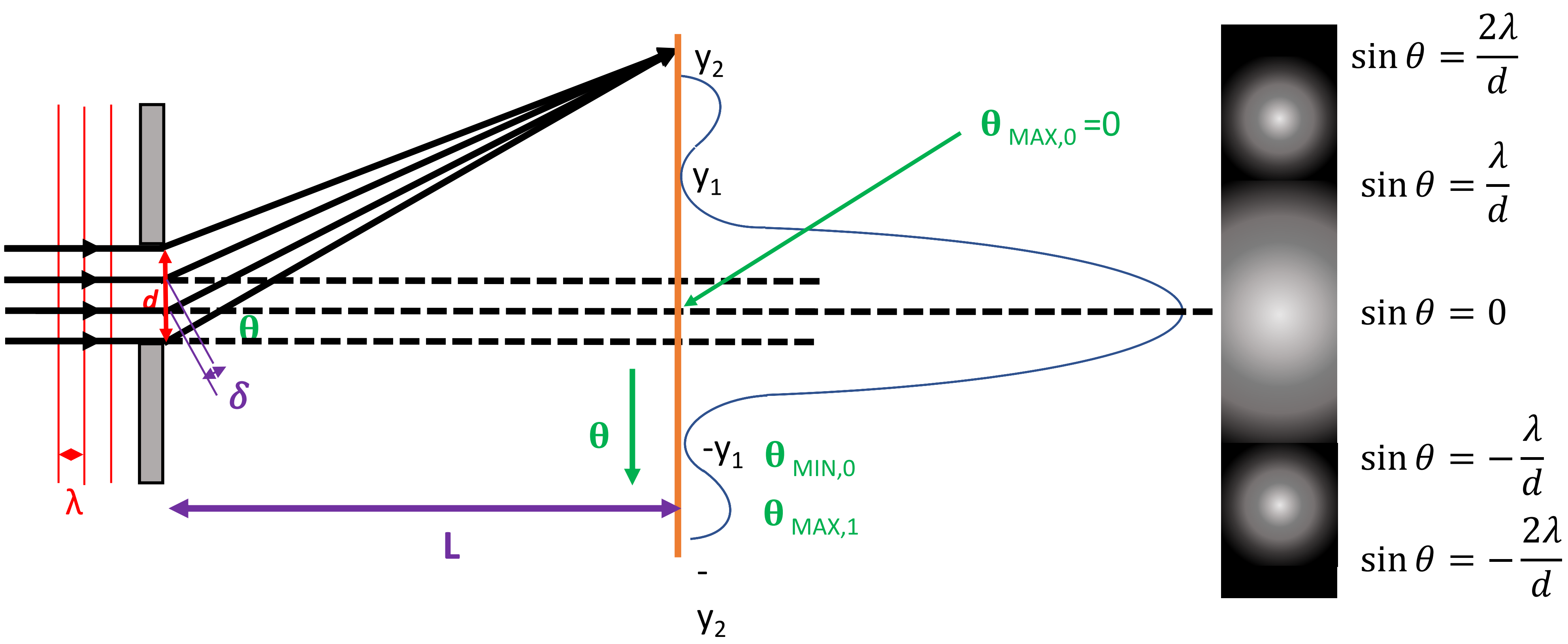
# Wave Optics: Diffraction

*No-one has ever been able to define the difference between interference and diffraction satisfactorily. It is just a question of usage, and there is no specific, important physical difference between them. The best we can do, roughly speaking, is to say that when there are only a few sources, say two, interfering, then the result is usually called interference, but if there is a large number of them, it seems that the word diffraction is more often used.*

– ***Richard Feynman.***

- As the wave travels through the slit, treat each infinitesimal point in the slit as a wave source.
- Light from one part can interfere with another part.
- Wavelet 1 has further to travel than 3.
- If additional distance $(\frac{d}{2}\sin\theta)$ equals λ/2 then wavelets from 1 and 3 are out of phase (destructive interference).
- Also true for any two points in the slit separated by d/2.

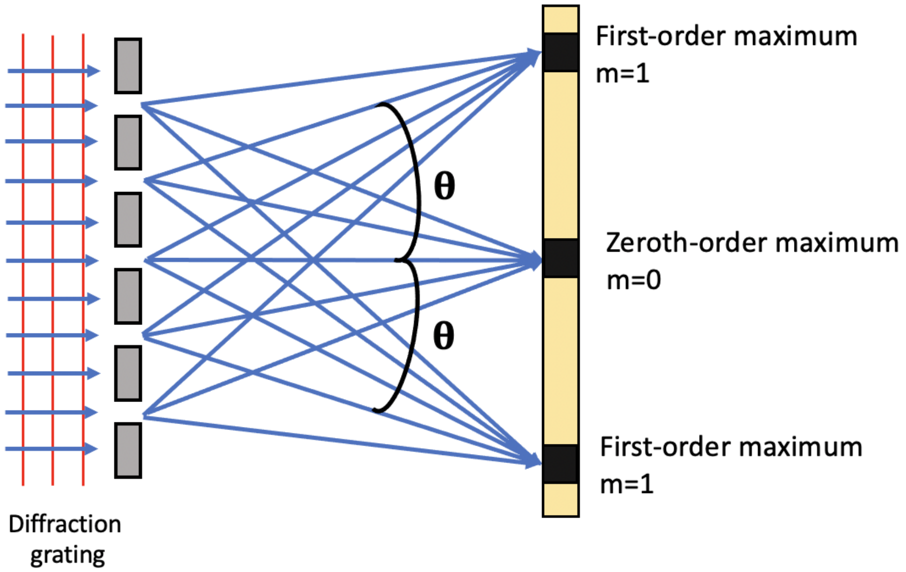


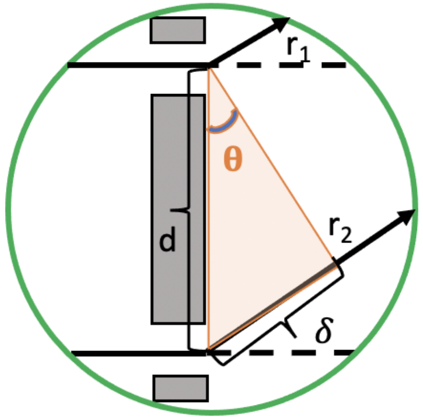



- Everything in phase at $\theta = 0 \;\rightarrow \textit{bright spot}$.
- Dark spots when: $\frac{d}{2}\sin\theta = \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots$
- Corresponding to: $\sin\theta_{1,2,3,\ldots} = \frac{\lambda}{d}, \frac{2\lambda}{d}, \frac{3\lambda}{d}, \ldots$

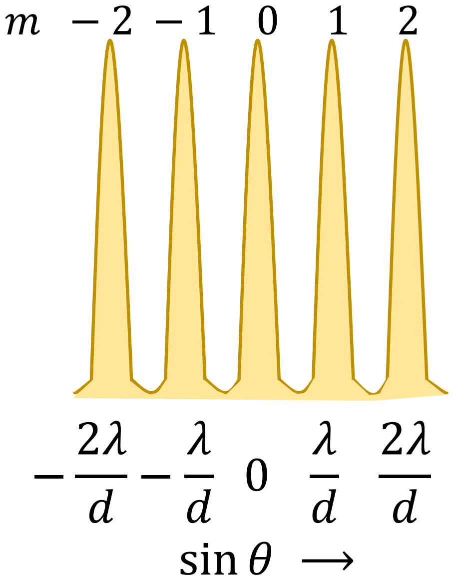
# Wave Optics: Diffraction

- Multiple slits is a diffraction grating.
- Interference between each diffracted wave.
- Path length difference between light passing through different slits.
  - Results in bright/darks spots.
- The more slits in the grating, the sharper the interference peaks.

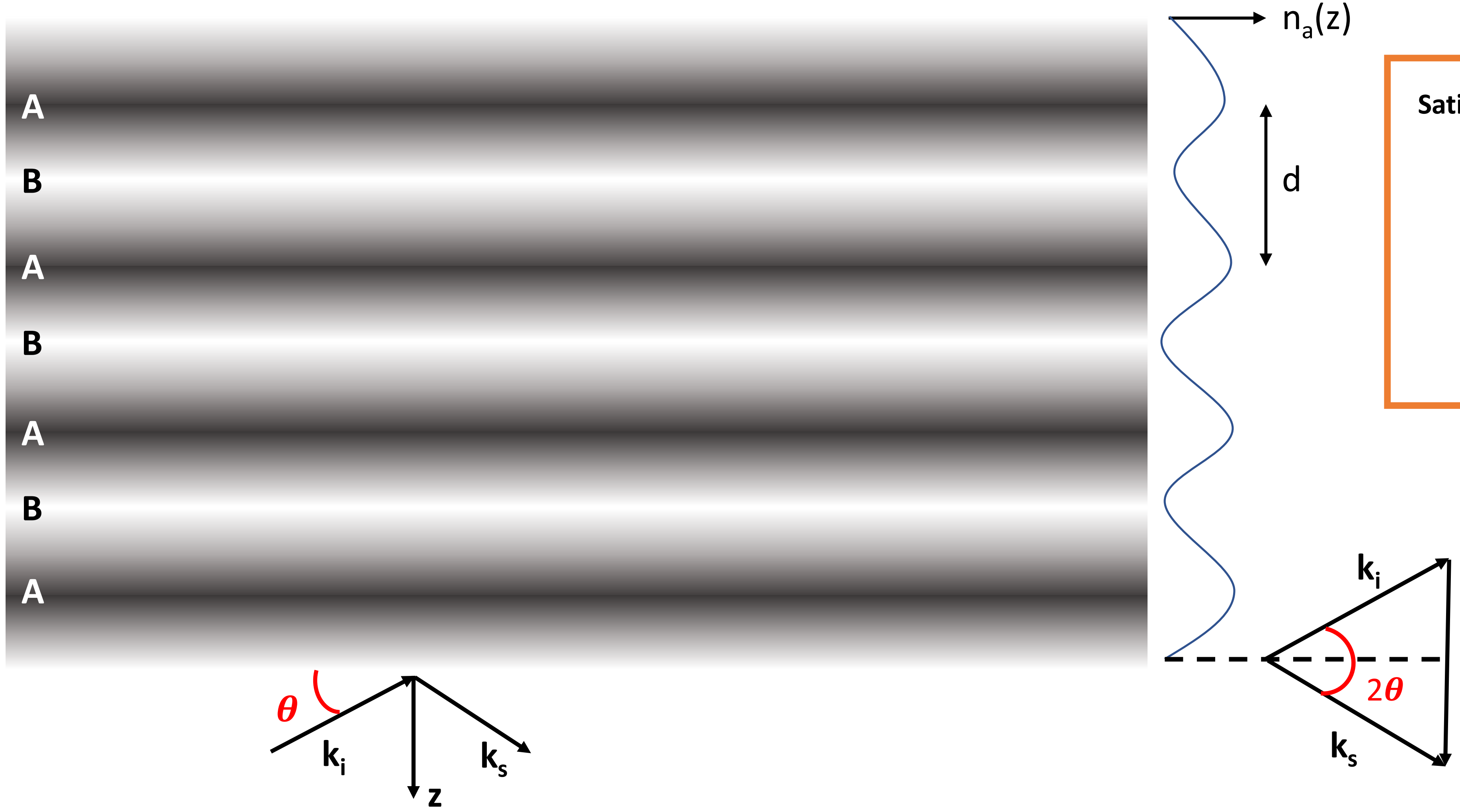



$$\delta = r_2 - r_1 = d \sin\theta$$



# Interference - *Multilayers*




**Satisfies Braggs Equation:**

$$\sin\theta = \frac{\frac{k_d}{2}}{k_i}$$

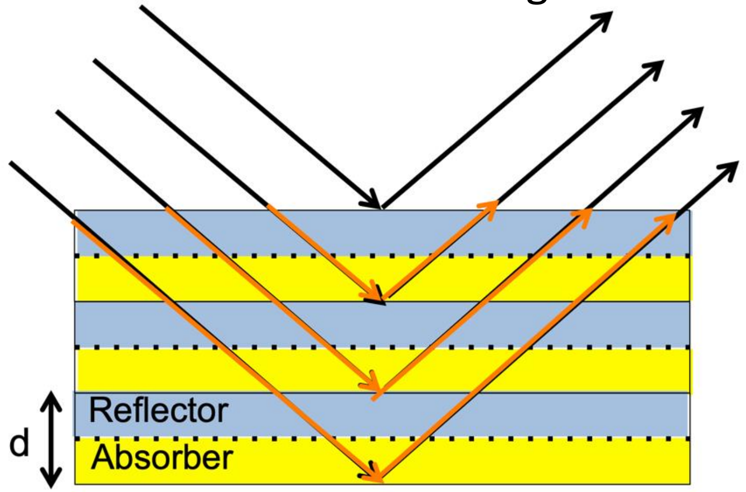
$$m\lambda = 2d\sin\theta$$

$$\boldsymbol{k_d} = \frac{2\pi}{d}\boldsymbol{z_0}$$

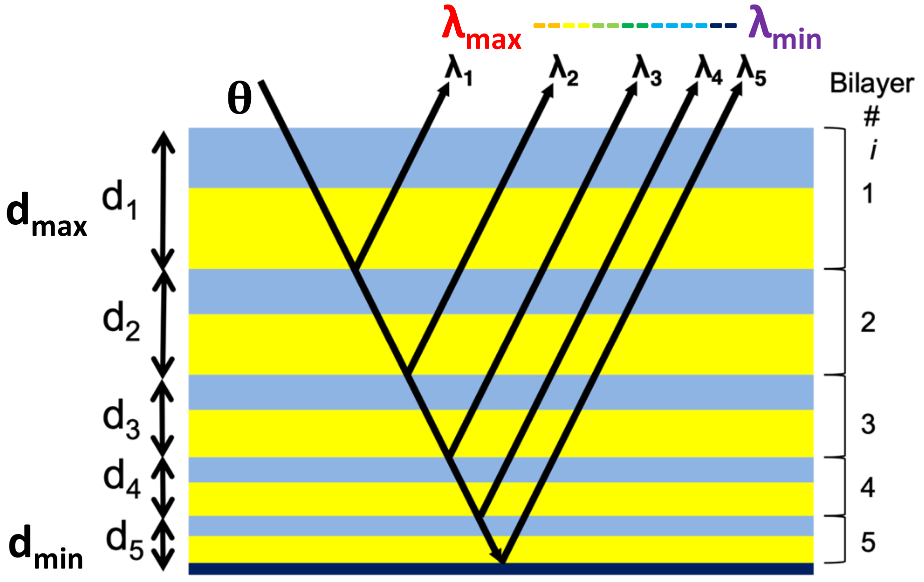
# Interference - *Multilayers*

- High reflectivity multilayer coatings require:
  - Change in refractive index at layer edges (sharp interface for scattering- absorption edge/glancing angle)
  - Minimal absorption in low-Z material
  - Thin high-Z layer (minimize absorption)
  - Chemical and thermal stability for each layer
  - Minimal roughness and diffusion at interfaces
  - No outgassing in vacuum
  - Uniform coating

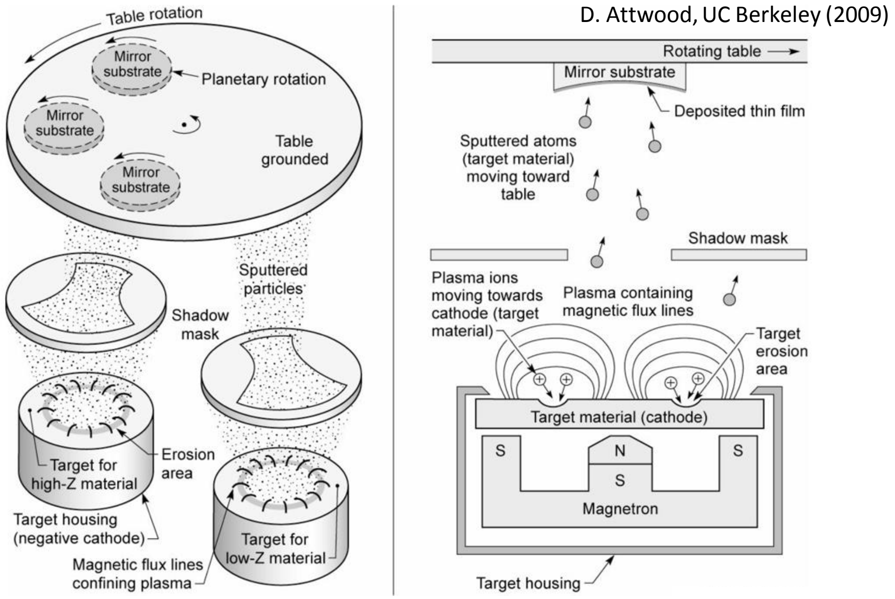
Periodic Grading

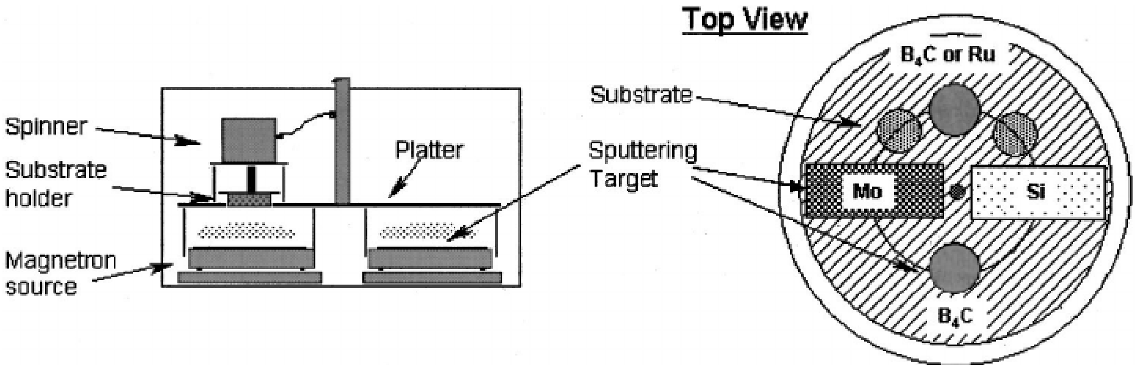



Non-periodic (depth) Grading



# Interference – Coating *Multilayers*




DC Magnetron sputtering reliably deposits coatings with excellent uniformity and predictable center wavelength

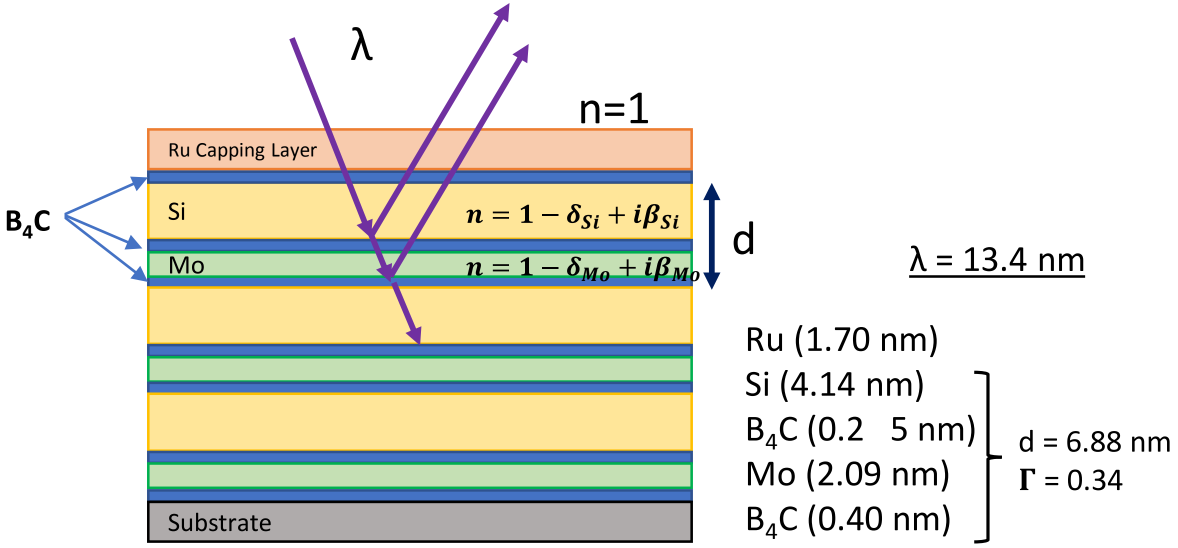
Changing table velocity provides precision control of radial thickness distribution and absolute film thickness

Substrate spins independently and quickly about its own axis for azimuthal uniformity.

Bajt, S. et al. (2001). Improved reflectance and stability of Mo/Si multilayers. Proc SPIE. 4506. 65-75. DOI: 10.1117/12.450946.

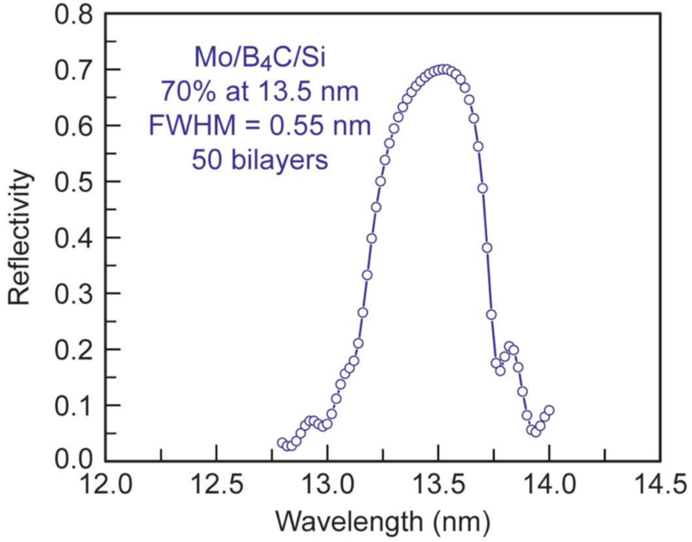
# Interference – Mo/Si

- Molybdenum as the "scattering layer"
- Silicon as the non-absorbing "spacer"
- Complex refractive index of each material plays a role in the reflection process
- Higher photon energies are absorbed in Si, lower photon energies (longer λ) are not.
- For Mo/Si, typically >13.4 nm (>92.5 eV)
- $\Gamma = \frac{\Delta t_{Mo}}{\Delta t_{Mo}+\Delta t_{Si}} = \frac{\Delta t_H}{d}$




$n_{Si} = 1 - 6.6 \times 10^{-5} + i\,1.8 \times 10^{-3}$

$n_{Mo} = 0.924 + i\,7.4 \times 10^{-3}$



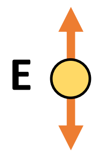

Bajt et al. (2001). Improved reflectance and stability of Mo/Si multilayers. Proc SPIE. 4506. 65-75. 10.1117/12.450946.

# Polarization

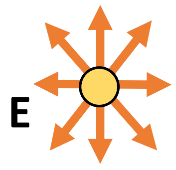
- Polarization is a property of a transverse wave (oscillation direction perpendicular to direction of motion), such as electromagnetic (EM) waves.

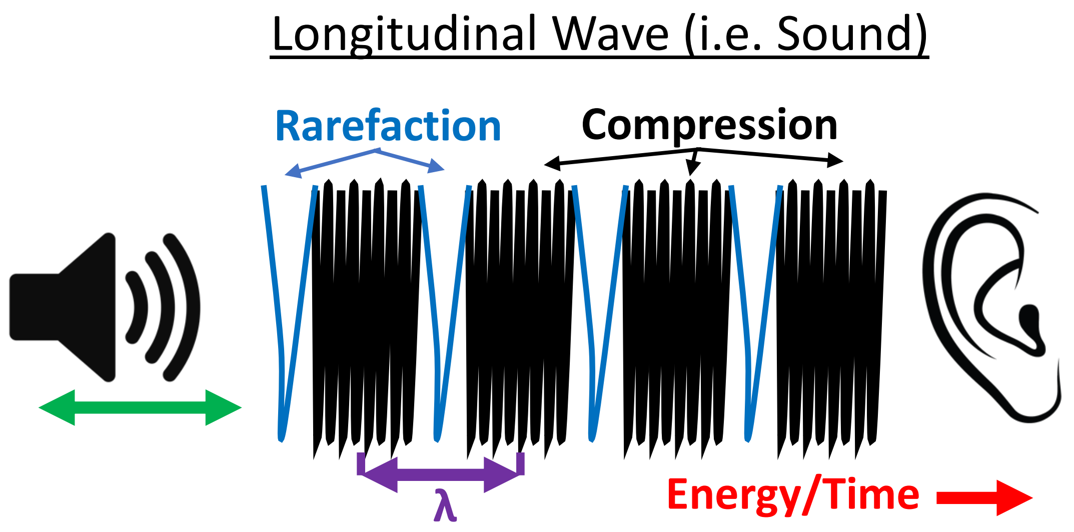
- EM waves consist of a coupled, oscillating electric field, always perpendicular to each other.

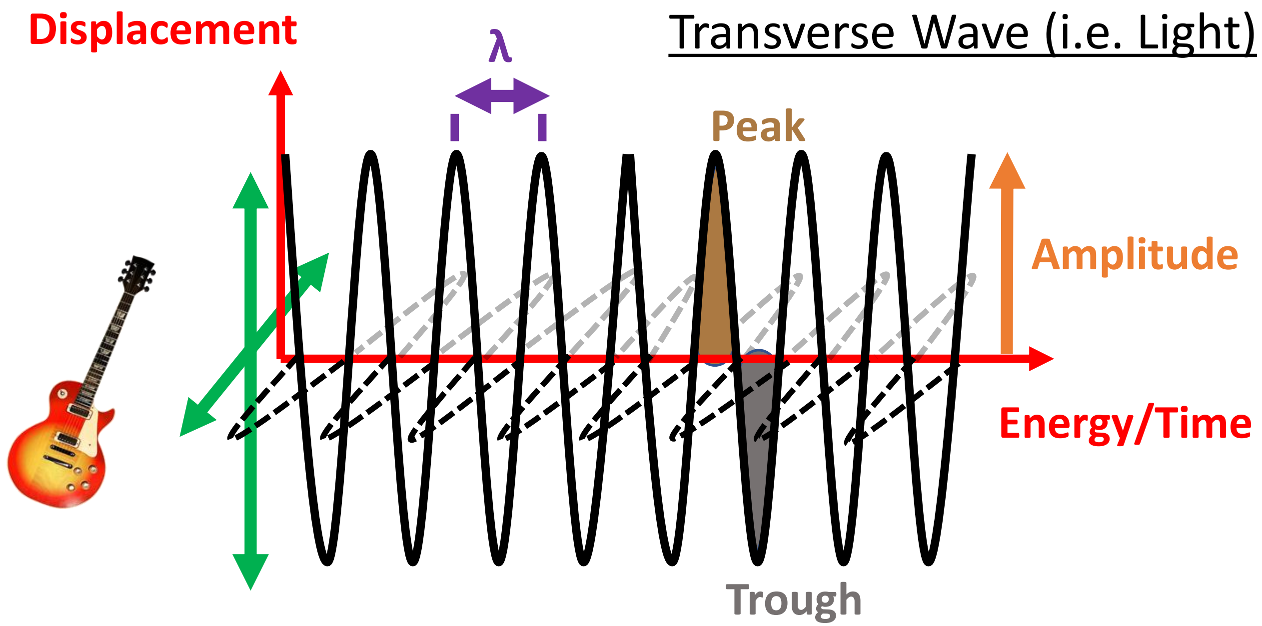
- **Polarized** EM waves: vibrations of the E-field lie along one single plane.



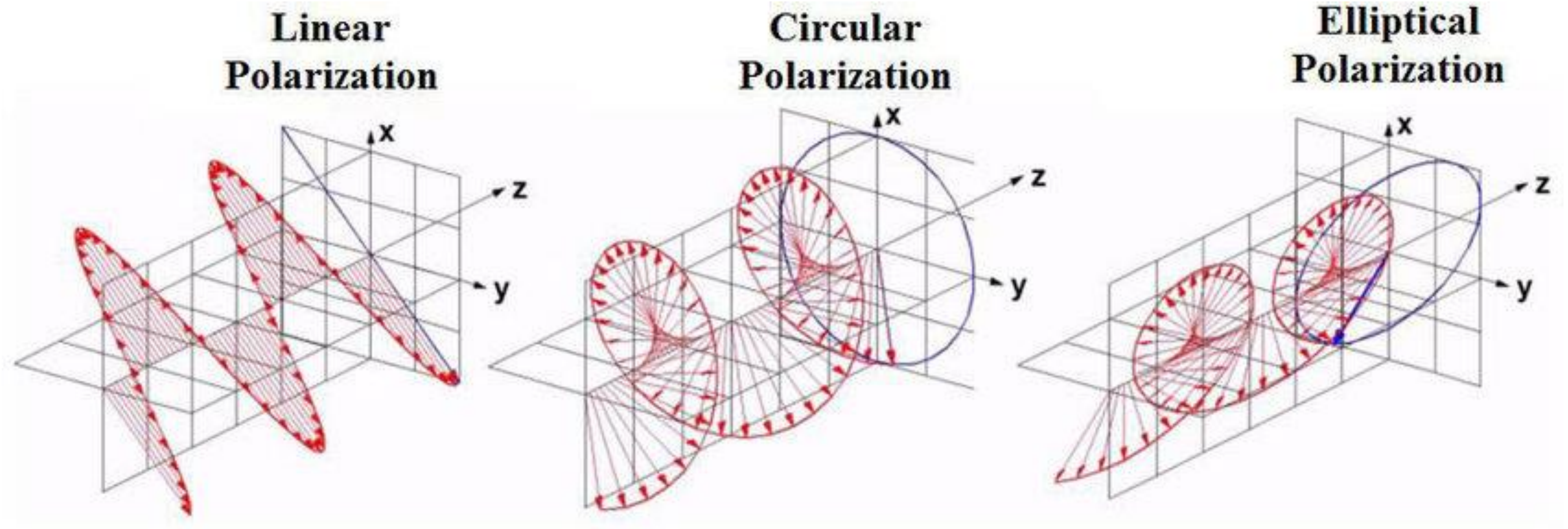

- **Unpolarized** EM waves: superposition of many beams in the same direction of propagation but each with *random* polarization.



# Polarization

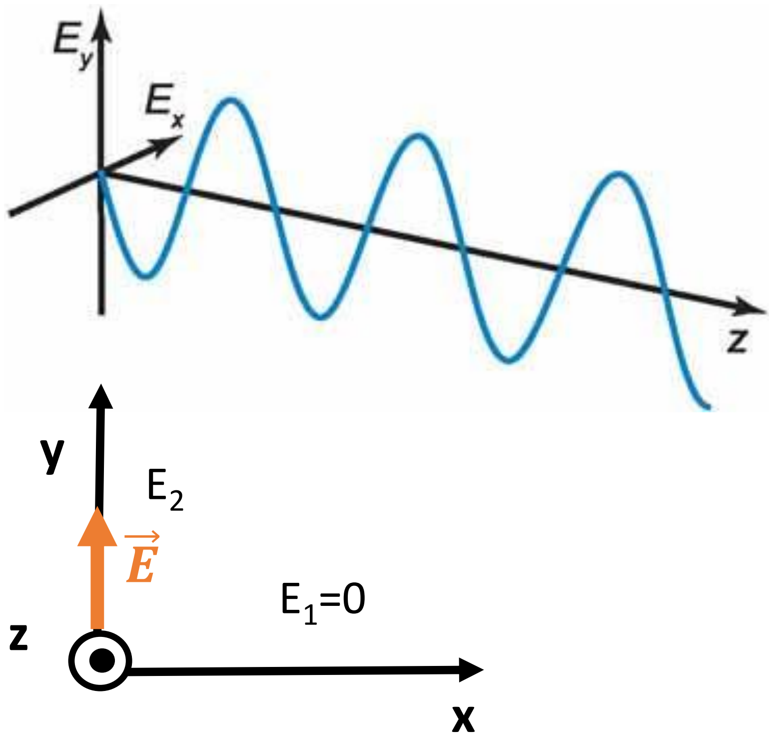



Credit: Siblini, Ali. (2017). Optimization of Antenna ARMA (Agile Matrix Antenna radiating by pixel elaborated with Meta-material) For beam forming for the RFID and Radar Applications.

# Linear Polarization

- Linear polarization
  - **Pane of the electric field doesn't change as the wave propagates.**
  - characterized by polarization angle (χ) and fraction (π)

Smith, D.G., (2013), https://doi.org/10.1117/3.883971.ch9




Consider a wave moving along the z-axis, and the electric field lies in the y-z plane. The wave is linearly polarized along the y-axis.

The electric field components along the x and y axes are:

$$E_x = 0$$

$$\boldsymbol{E_y = E_2 \cos(\omega t - \beta z)}$$

where, $\omega = 2\pi\nu$ (angular frequency) and $\beta = \frac{2\pi}{\lambda}$ (wave number)

# Linear Polarization

- Electric field vector **E** is the vector sum of the $\mathbf{E_x}$ and $\mathbf{E_y}$ components:

$$\vec{E} = \overrightarrow{\boldsymbol{E}_x} + \overrightarrow{E_y}$$

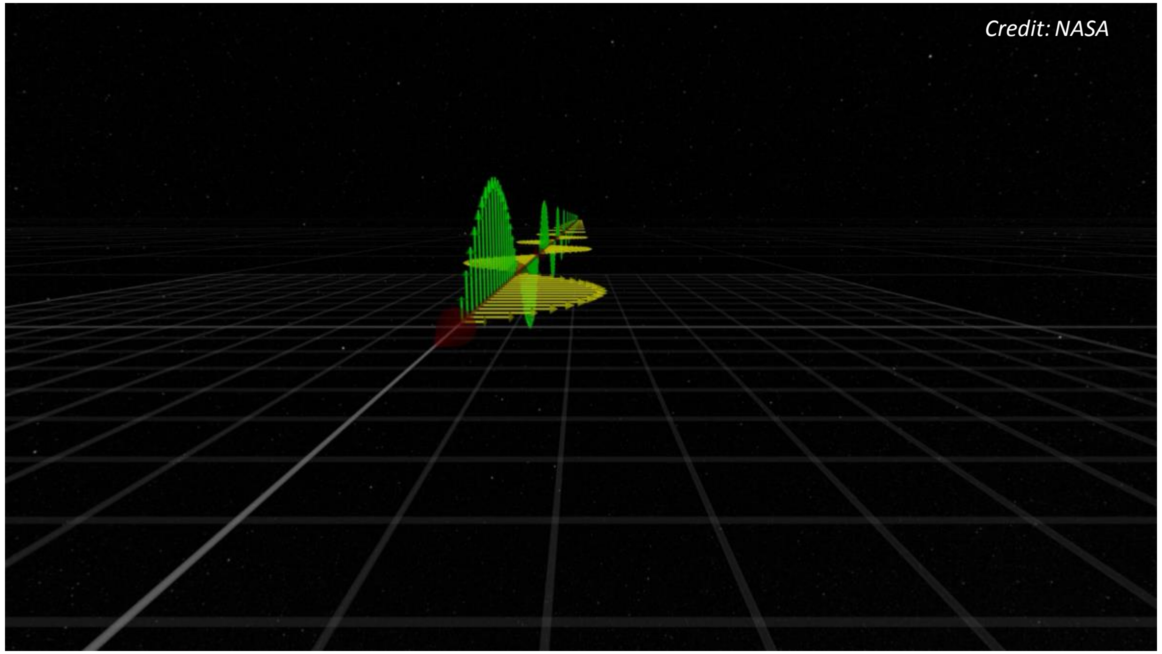
where,

$$E_x = E_1 \sin(\omega t - \beta z)$$
$$E_y = E_2 \sin(\omega t - \beta z)$$

- Consider a case where **E** lies in a plane between *y-z* and x-z at inclination α, w.r.t to the x-axis. $E_x$ and $E_y$ become:

$$E_x = E \cos(\alpha)$$
$$E_y = E \sin(\alpha)$$

- The ratio of $E_y$ to $E_x$ is: $$\frac{E_y}{E_x} = \tan\alpha$$



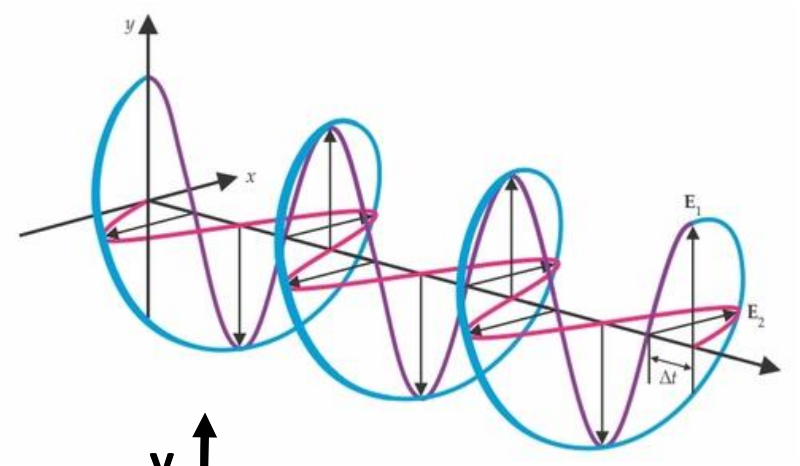

Green: Magnetic field
Yellow: Electric field

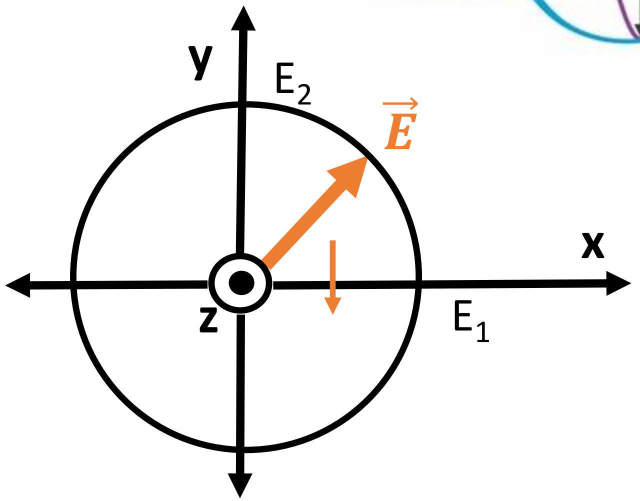
NB: For linear polarization:
- Phase difference between $E_x$ and $E_y$ = 0
- Magnitudes of $E_x$ and $E_y$ can be different

# Circular Polarization

- If magnitudes of $E_x$ and $E_y$ are the same, but there is a phase difference of $\frac{\pi}{2}$ or $-\frac{\pi}{2}$, the tip of $\vec{E}$ vector traces a circle. Wave is circularly polarized (either right-handed or left-handed ($E_y$ is -ve).

Miller, J.L., Physics Today **72**, 4, 17 (2019); https://doi.org/10.1063/PT.3.4178




Phase shift of $\frac{\lambda}{4}$. Quarter-wave plate converts polarization from linear to circular and vice versa. Electric fields expressed as:

$$E_x(z,t) = E_1 \sin(\omega t - \beta z)$$

$$E_y(z,t) = E_2 \sin(\omega t - \beta z + \delta), \qquad \delta = \pm\frac{\pi}{2}$$

$$\rightarrow E(z,t) = E_2 \cos(\omega t - \beta z)$$

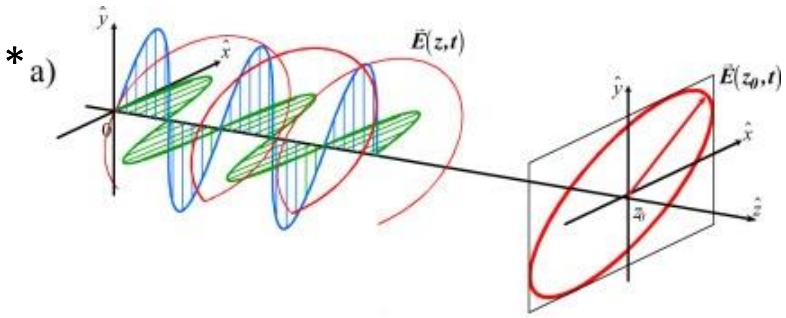
$$Axial\ ratio = \frac{Major\ axis}{Minor\ axis} = \frac{|E_y|}{|E_x|} = \frac{E_2}{E_1} = 1$$

NB: Axial ratio always unity

$$\therefore\ \vec{E} = \overrightarrow{e_x} E_1 \sin(\omega t - \beta z) + \overrightarrow{e_y} E_2 \sin\left(\omega t - \beta z \pm \frac{\pi}{2}\right)$$

# Elliptical Polarization

- If the magnitudes of Ex and Ey are **not** equal, **and** there is a phase difference between the two components, the tip of $\vec{E}$ maps an ellipse. *The wave is elliptically polarized* (linear + circular polarization).



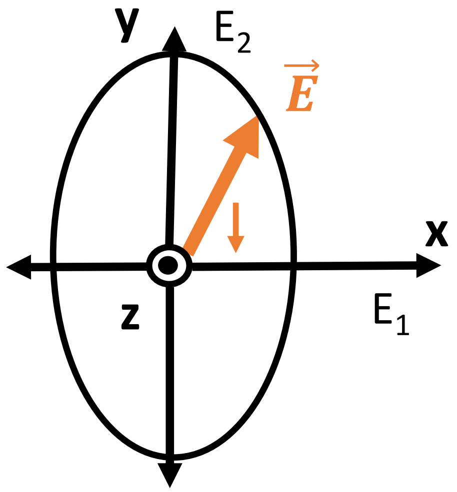

The electric fields are:

$$E_x = E_1 \sin(\omega t - \beta z)$$

$$E_y(z,t) = E_2 \sin(\omega t - \beta z + \delta), \qquad \delta \neq 0$$

$$Axial\ ratio = \frac{Major\ axis}{Minor\ axis} = \frac{|E_y|}{|E_x|} = \frac{E_2}{E_1} \neq 1$$

NB: $|E_y| \neq |E_x|, \therefore E_2 \neq E_1$



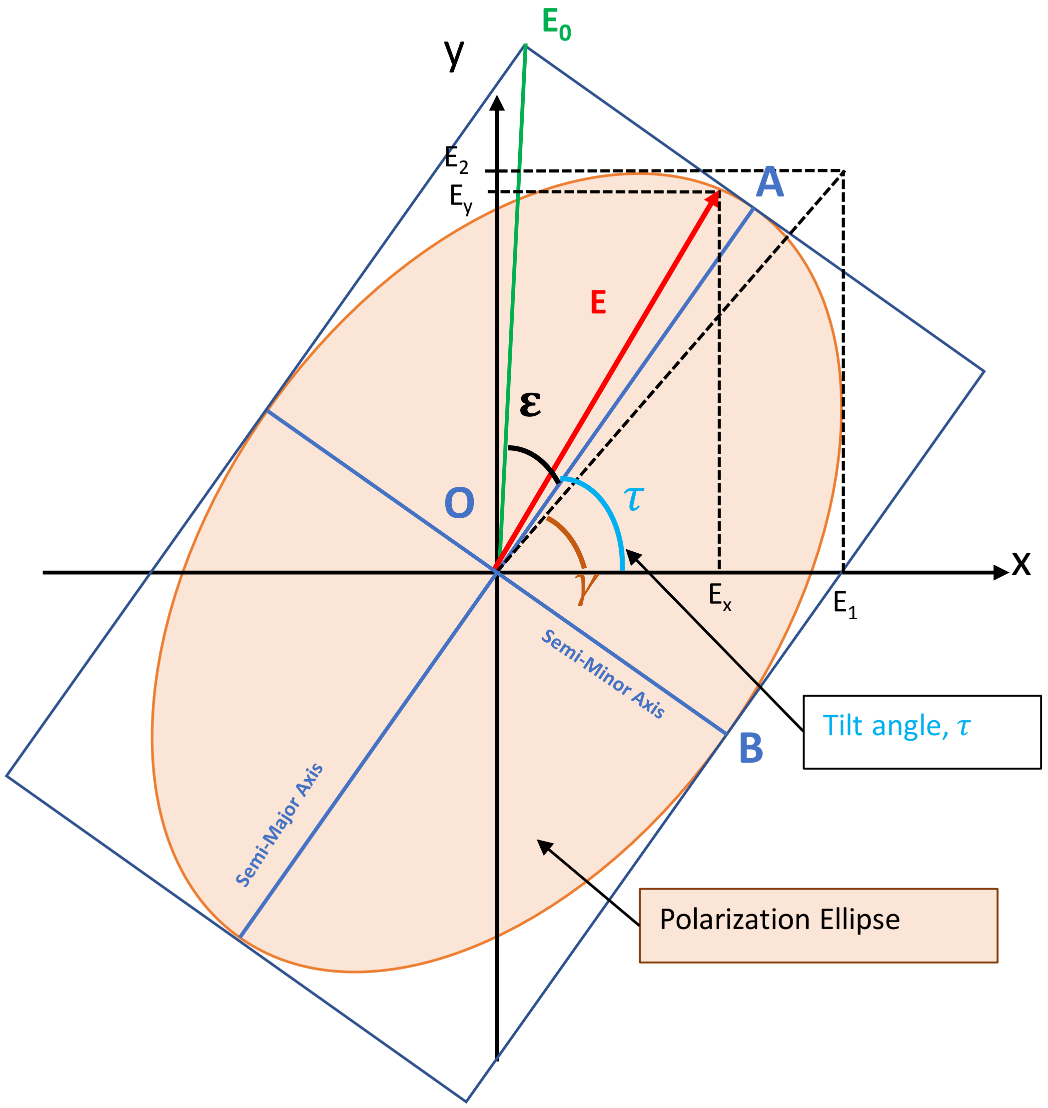

$$\therefore\ \vec{E} = e_x E_1 \sin(\omega t - \beta z) + e_y E_2 \sin(\omega t - \beta z + \delta)$$

Circular polarization is similar to elliptical polarization, where $E_2 = E_1, \delta = \pm\frac{\pi}{2}$ and the axial ratio is unity.

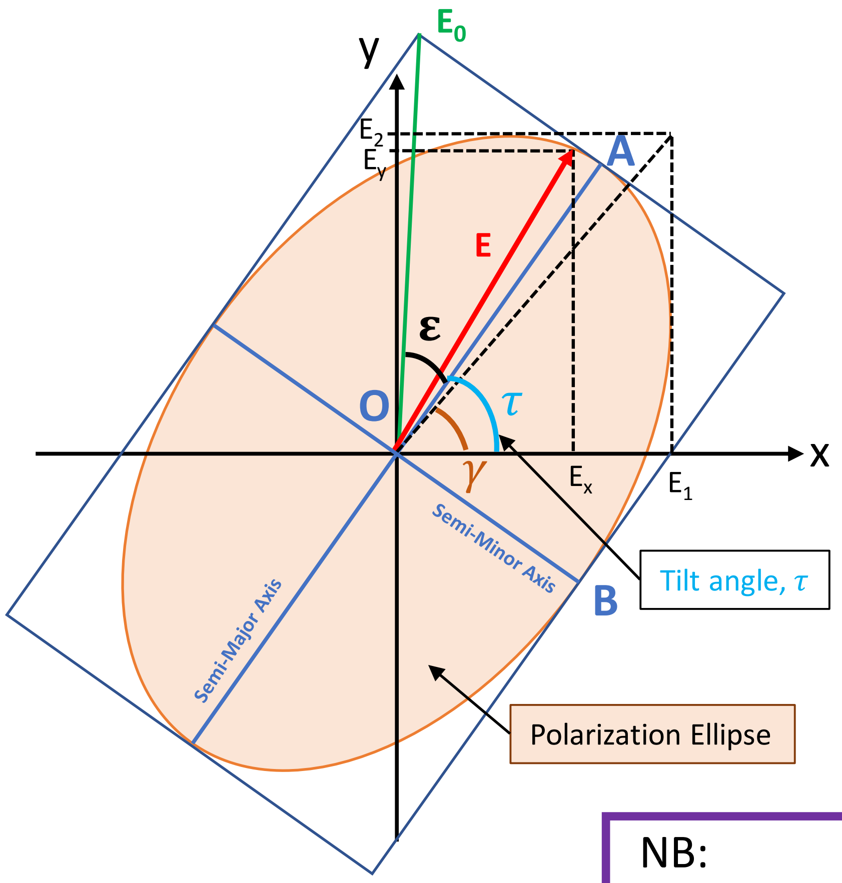
* Image Credit: Laurent Ferro-Famil, Eric Pottier, Microwave Remote Sensing of Land Surface, Figure 2a, 2016, Elsvier

# Polarization Ellipse

- Any complete polarization resulting from a coherent source can be analysed using a polarization ellipse, drawn by tip of E-Field.
- $\chi\ or\ \varepsilon$ is the angle of ellipticity
- $\psi\ or\ \tau$ is the tilt or orientation angle



# Polarization Ellipse




<u>Generic form of **E**:</u>

$$\vec{E} = e_x E_1 \sin(\omega t - \beta z) + e_y E_2 \sin(\omega t - \beta z + \delta)$$

<u>At z=0 in x-y plane:</u>

$$E_x = E_1 \sin(\omega t) \quad \boldsymbol{or} \quad \sin \omega t = \frac{E_x}{E_1}$$

$$E_y = E_2 \sin(\omega t + \delta) \quad \boldsymbol{or} \quad \sin \omega t \cos \delta + \cos \omega t \sin \delta \; = \frac{E_y}{E_2}$$

<u>Substituting:</u>

$$\frac{E_y}{E_2} = \frac{E_x}{E_1} \cos \delta + \sqrt{1 - \left(\frac{E_x}{E_1}\right)^2} \sin \delta \quad \rightarrow \sqrt{1 - \left(\frac{E_x}{E_1}\right)^2} \sin \delta = \frac{E_y}{E_2} - \frac{E_x}{E_1} \cos \delta$$

NB:

$\omega = 2\pi\nu \quad \beta = \dfrac{2\pi}{\lambda}$

$\delta$=phase shift

<u>Simplify and square:</u>

$$\left(\frac{E_x}{E_1}\right)^2 - 2\left(\frac{E_x}{E_1}\right)\left(\frac{E_y}{E_2}\right) \cos \delta + \left(\frac{E_y}{E_2}\right)^2 = sin^2\delta \longrightarrow$$

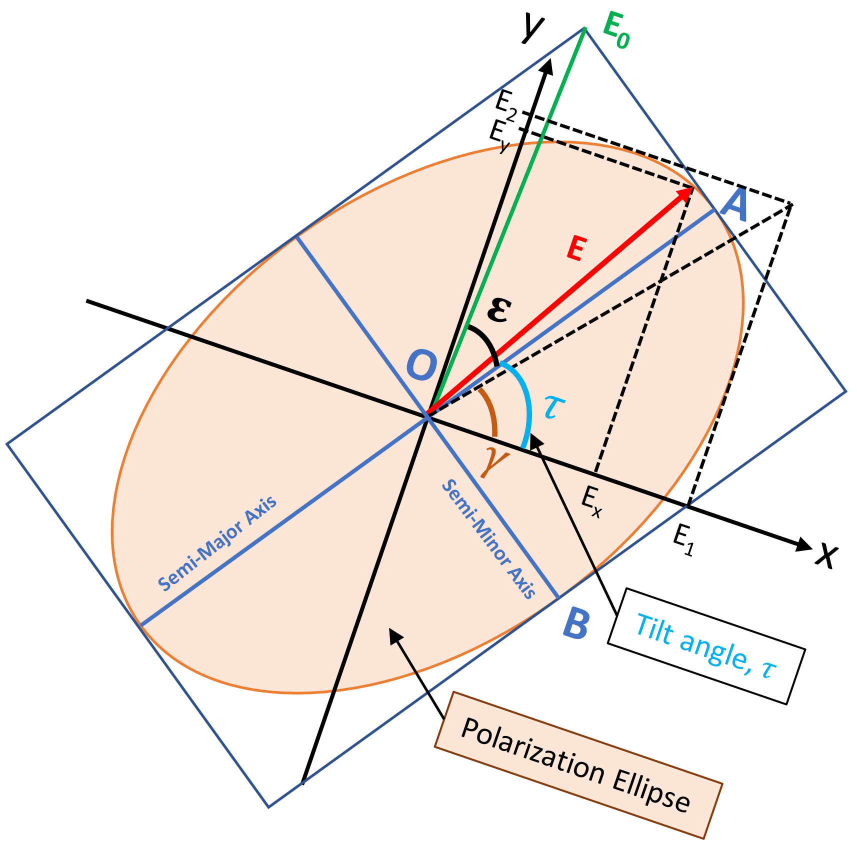
Can be rewritten:

$$aE_x^2 + bE_xE_y + cE_y^2 = 1, \qquad \text{where:}$$

$$a = \frac{1}{E_1^2 sin^2\delta}, \quad b = \frac{2\cos\delta}{E_1E_2 sin^2\delta}, \quad c = \frac{1}{E_2^2 sin^2\delta},$$

# Polarization Ellipse

- Any complete polarization resulting from a coherent source can be analysed using a polarization ellipse, drawn by tip of E-Field.
- $\chi$ or $\varepsilon$ is the angle of ellipticity
- $\psi$ or $\tau$ is the tilt or orientation angle

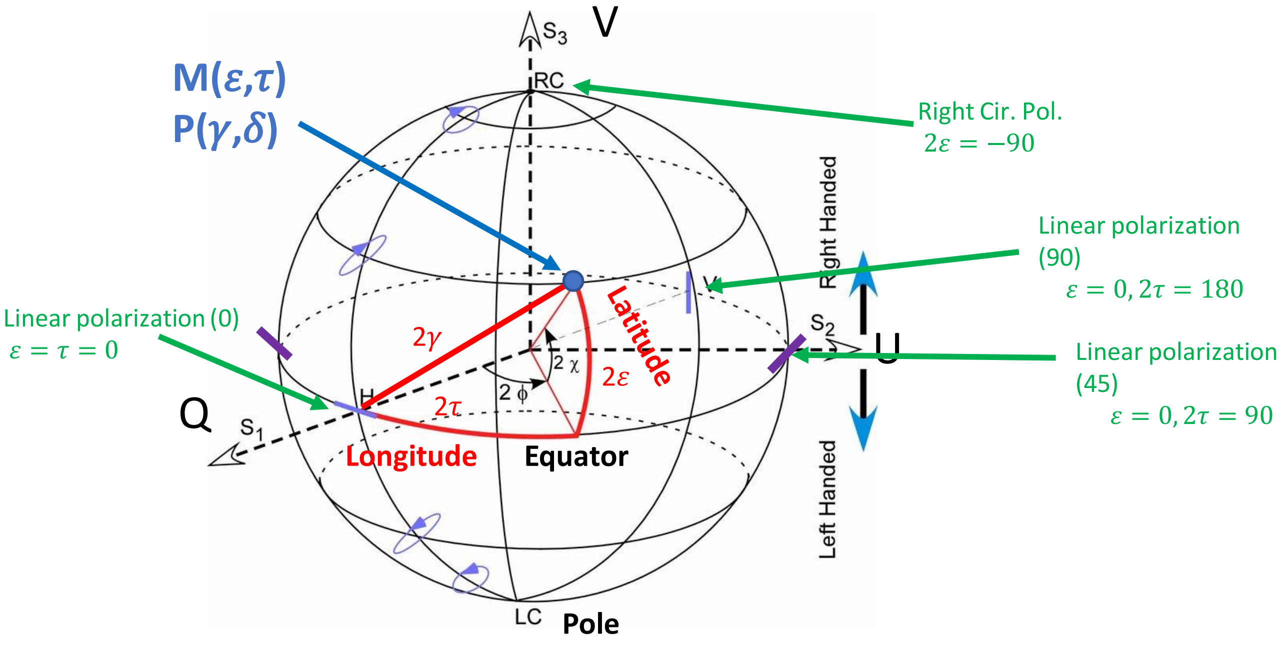



$$Axial\ Ratio\ (AR) = \frac{OA}{OB} = \frac{|E|\sqrt{\frac{1+\sqrt{1-sin^2(2\theta)sin^2\beta}}{2}}}{\sqrt{\frac{1-\sqrt{1-sin^2(2\theta)sin^2\beta}}{2}}}, \qquad 1 \le AR \le \infty$$

$$\gamma = \arctan\left(\frac{E_2}{E_1}\right), 0° \le \gamma \le 90° \qquad \varepsilon = \text{arccot}(\pm AR), -45° \le \varepsilon \le +45°$$

# Poincare Sphere

- Spherical surface occupied by completely polarized states, where the longitude and latitude are related to the polarization ellipse.
- $Ip, 2\tau$ and $2\varepsilon$ are spherical coordinates of the 3D vector of cartesian coordinates ($S_1$, $S_2$, $S_3$).
- I is the total beam intensity
- p is the polarization degree




Adapted from Ainouz, Samia et al.. (2006).
Optics express. 14. 5916-27. 10.1364/OE.14.005916.

$$S_0 = I$$
$$S_1 = Ip \cos 2\tau \cos 2\varepsilon$$
$$S_2 = Ip \sin 2\tau \cos 2\varepsilon$$
$$S_3 = Ip \sin 2\varepsilon$$

$$p = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0}$$

$$2\varepsilon = \arctan \frac{S_2}{S_1} \qquad 2\tau = \arctan \frac{S_3}{\sqrt{S_1^2 + S_2^2}}$$

Longitude = $2\tau$
Latitude = $2\varepsilon$
Great circle angle = $2\gamma$
Equator to GCA = $\delta$

# Polarization: Stokes Parameters

- Stokes Parameters describe the polarization state of EM radiation in terms of total intensity (I), degree of polarization (p) and the shape parameters of the polarization ellipse.

- For a classical EM wave that is 100% polarized over short time intervals during the wave period (but whose polarization changes on longer timescales), the Stokes parameters are defined by time averages of the E-field strength. Assuming wave propagation along the z-axis:

$$S_0 = I = \langle E_x^2 + E_y^2 \rangle$$
$$S_1 = Q = \langle E_x^2 - E_y^2 \rangle$$
$$S_2 = U = \langle 2E_x E_y \cos\delta \rangle$$
$$S_3 = V = \langle 2E_x E_y \sin\delta \rangle$$

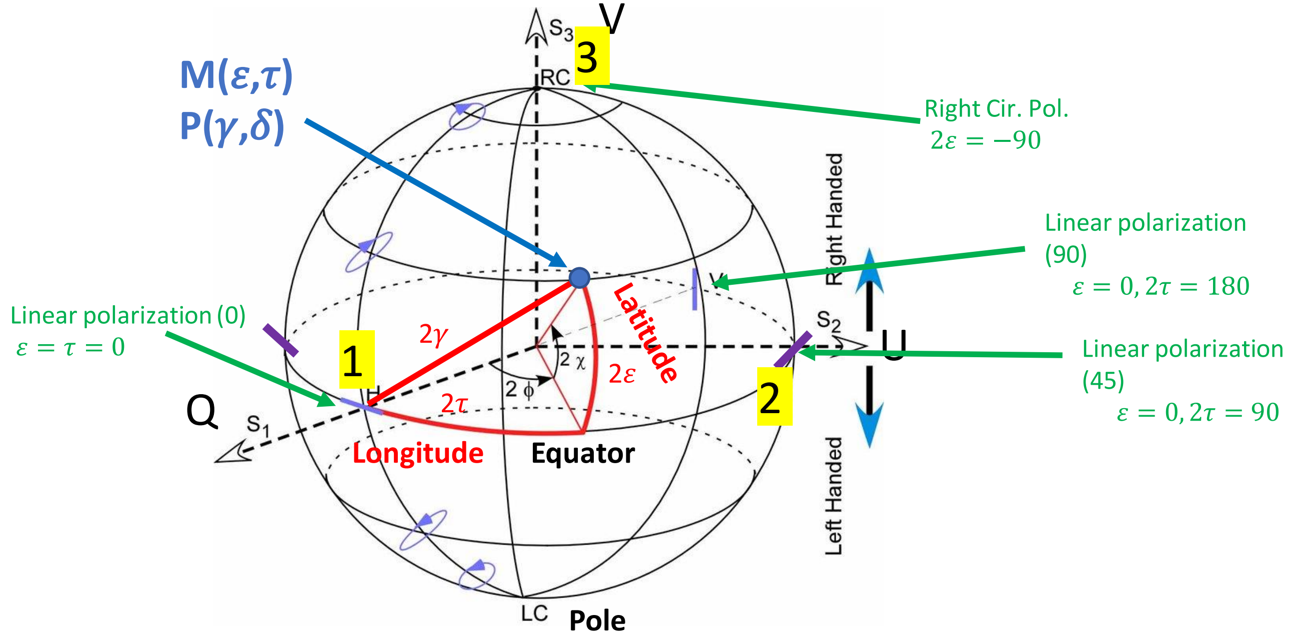
I is the Intensity/flux of the wave
Q and U depend on the linear polarization properties
V depends on circular polarization properties

- $E_x$ ($E_y$) is proportional to the E-field along the x-axis (y-axis), and $\delta$ is the lag of $E_y$ behind $E_x$.

# Stokes Parameters

Adapted from Ainouz, Samia et al.. (2006). Optics express. 14. 5916-27. 10.1364/OE.14.005916.

- For a 100% *linear* polarized wave with an **E**-field vector **along the x-axis** (y-axis):
  - Q = I (-I)
- For a 100% *linear* polarized wave with an **E**-field vector **along the diagonal** between the x- and y-axes (negative x- and y-axes ):
  - U = I (-I)
- For a 100% ***circularly*** right-handed (left-handed) polarized:
  - V = I (-I)

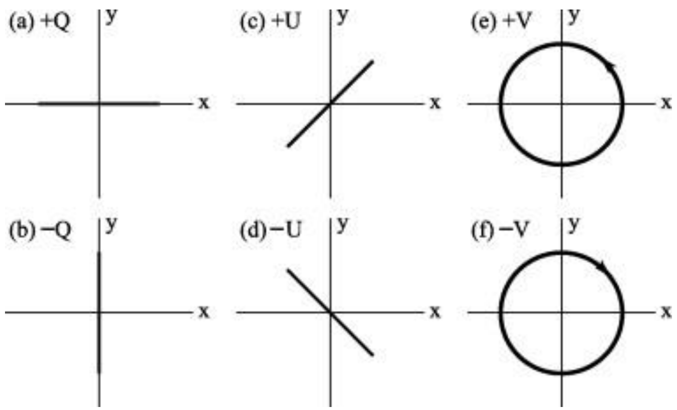



$$\vec{S} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} \overset{1}{=} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ LP (Hori.)}$$

$$\overset{3}{=} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ RHCP}$$

$$\overset{2}{=} \qquad \text{LP (+45°)}$$

(a) Q>0, U=0, V=0
(b) Q<0, U=0, V=0
(c) U>0, Q=0, V=0
(d) U<0, Q=0, V=0
(e) V>0, Q=0, U=0
(f) V<0, Q=0, U=0

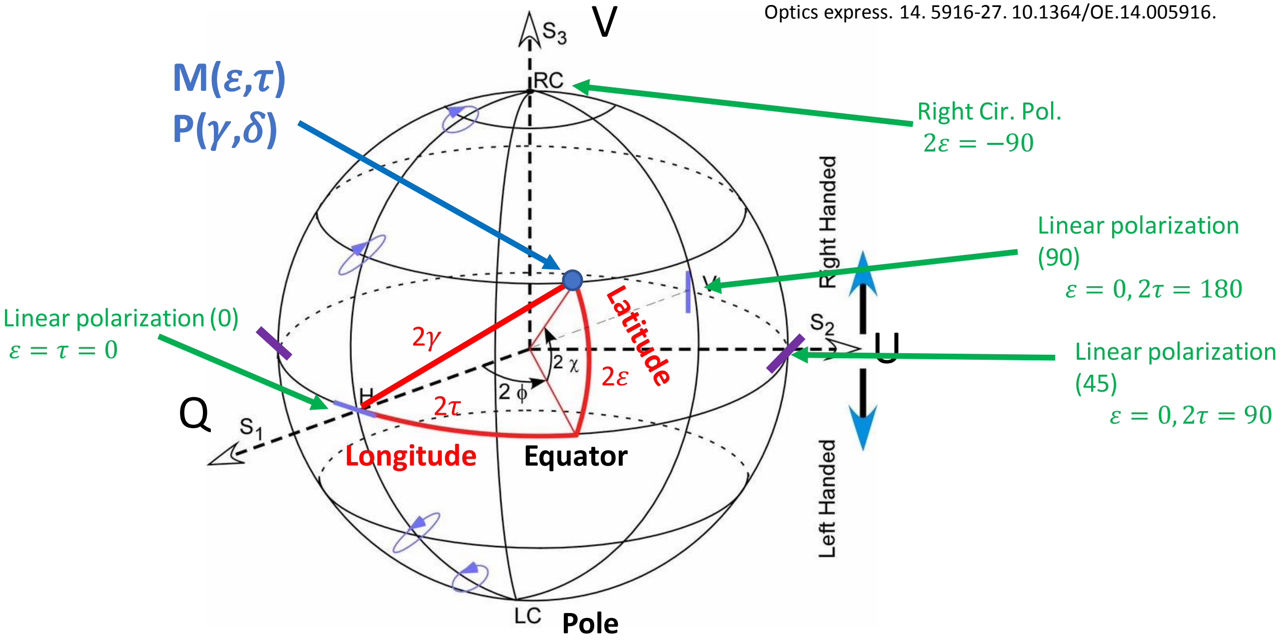



F. Kislat, et al., (2015) Astroparticle Physics, 68, 45-51 https://doi.org/10.1016/j.astropartphys.2015.02.007 .

# Stokes Parameters

- For a 100% polarized wave:
  - $I = \sqrt{Q^2 + U^2 + V^2}$
- For a linearly polarized wave (V=0):
  - $I = \sqrt{Q^2 + U^2}$
- Q and U given by the polarization direction $\psi$ (angle between the x-axis and **E**-field direction) by:
  - $Q = \cos 2\tau$
  - $U = \sin 2\tau$
- Implying:
  - $\tan 2\tau = \frac{U}{Q}$
- Polarization fraction (polarized wave flux):
  - $p = \frac{\sqrt{Q^2+U^2+V^2}}{I}$
- For V=0, linear polarization fraction is:
  - $p = \frac{\sqrt{Q^2+U^2}}{I}$

Adapted from Ainouz, Samia et al.. (2006). Optics express. 14. 5916-27. 10.1364/OE.14.005916.




$$S_0 = I = \langle E_x^2 + E_y^2 \rangle$$
$$S_1 = Q = \langle E_x^2 - E_y^2 \rangle$$
$$S_2 = U = \langle 2E_xE_y \cos\delta \rangle$$
$$S_3 = V = \langle 2E_xE_y \sin\delta \rangle$$

# Polarization

- Fit function:

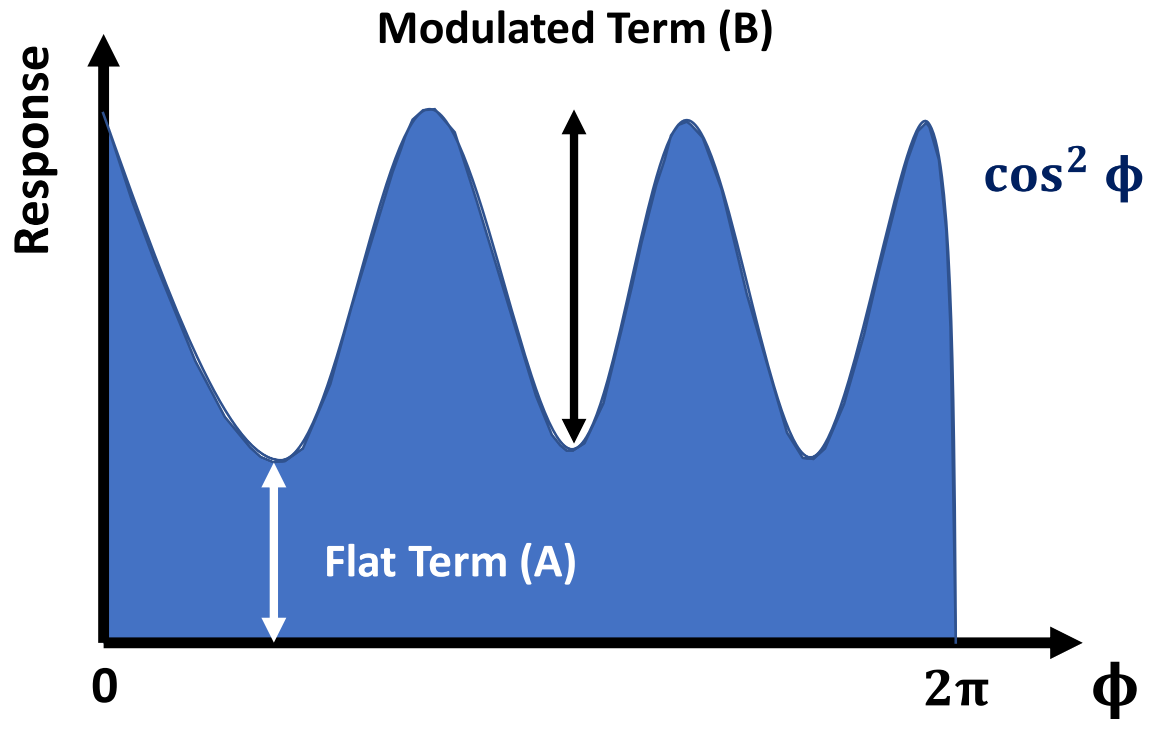
$$M(\phi) = A + B\, cos^2(\phi - \phi_0)$$

- Modulation:

$$\frac{M_{max} - M_{min}}{M_{max} - M_{min}} = \frac{B}{B + 2A}$$

- Polarization:

$$\frac{1}{\mu}\frac{B}{B + 2A}$$




μ is the modulation factor, i.e. the modulation for 100% polarized radiation

# Polarization: MDP

- Polarimeter deals with counting rate statistics
  - mainly depends on the modulation factor, μ (response of a polarimeter to a 100% polarized source)

- Minimum Detectable Polarization (MDP) at 99% confidence level:

$$MDP_{99}(\%) = \left[\frac{4.29}{\mu\, R_s}\right] \times \left[\frac{(R_S + R_B)}{(t)}\right]^{1/2}$$

where M is the variation in the signal as a function of azimuthal angle produced by a 100% polarized signal with $R_B$ =0, $R_B$ is the background count rate. $R_S$ is the source count rate and t is the integration time.

If the background is negligible then: $MDP = \frac{4.29}{\mu\sqrt{N_{ph}}}$

To reach MDP=1% with μ=0.5: $N_{ph} = \left(\frac{4.29}{(\mu \times MDP)}\right)^2 = 7.36 \times 10^5\, ph$

Source detection > 10 photons
Source spectral slope > 100 photons
Source polarization > 100,000 photons

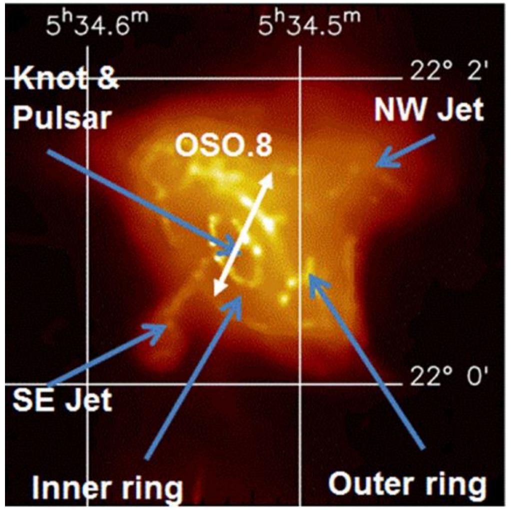
NB: confidence level equivalent is 4.29 → 7.58 for 5σ Gaussian.

# X-ray Polarization (Astronomy)

- **1972**: *First* astronomical X-ray polarization measurement (Aerobee 350 rocket, Crab Nebula)
  - p=15 ± 5%, $\theta$=156 ± 10°

- **1978**: *Last* astronomical X-ray polarization measurement (8th Orbiting Solar Observatory, Crab Nebula)
  - Highly significant
  - p=19 ± 1%, $\theta$=156.4 ± 1.4° (2.6 keV)

- **2021/2**: IXPE (many targets)
  - Imaging capabilities will allow to measure pulsar polarization from nebula



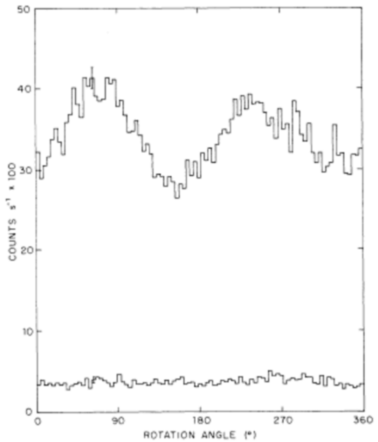




FIG. 2.—Average modulation curves obtained with both detectors at 2.6 keV during (*upper curve*) observations of the Crab Nebula and during (*lower curve*) observations of the Earth-occulted instrumental background.

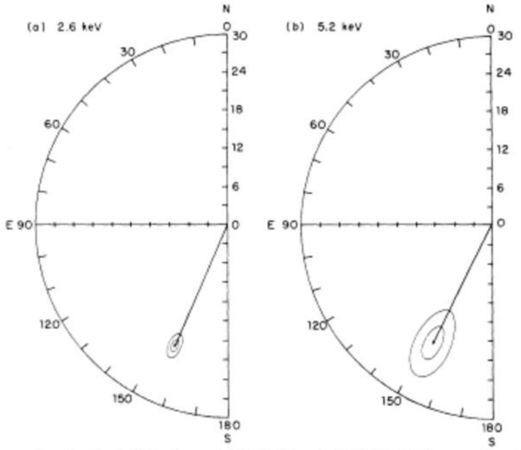


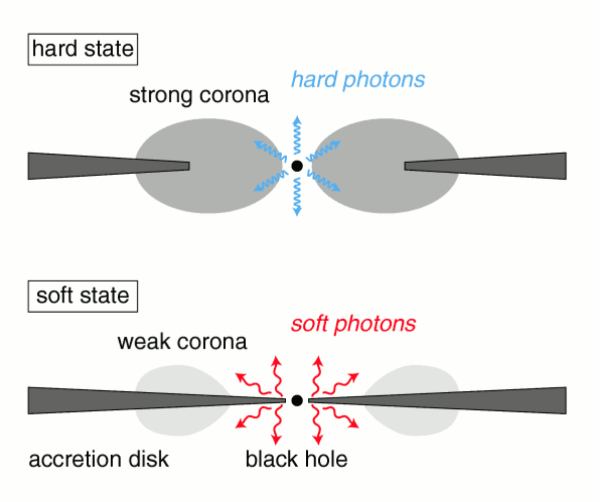

FIG. 3.—The polarization vectors for the Crab Nebula at (*a*) 2.6 keV and (*b*) 5.2 keV. Surrounding the vectors in order of increasing size are the 67% and 99% confidence contours. The radial scale is the polarization in percent.

Weisskopf, M.C. et al., (1978), **220**, L117-121; 10.1086/182648

# X-ray Polarization (Astronomy)

- Theoretical X-ray polarization
  - Cyclotron (Rees, 1975)
  - Synchrotron (Westfold, 1959)
  - Non-thermal Bremsstrahlung (Brown, 1971)
  - Scattering (Sunyaev & Titarchuk, 1985)
  - General Relativity (Stark & Connors, 1977)
  - Magnetic fields (Gnedin & Sunyaev, 1974)

- X-ray polarization highly sensitive tool for:
  - Morphology of the source
  - geometry of (re-)accreting material
  - magnetic field strength
  - Spacetime and X-ray propagation




Meyer-Hofmeister, E. et al. A&A, 2005, 431:1, 181-187

# Polarization - Instrumentation

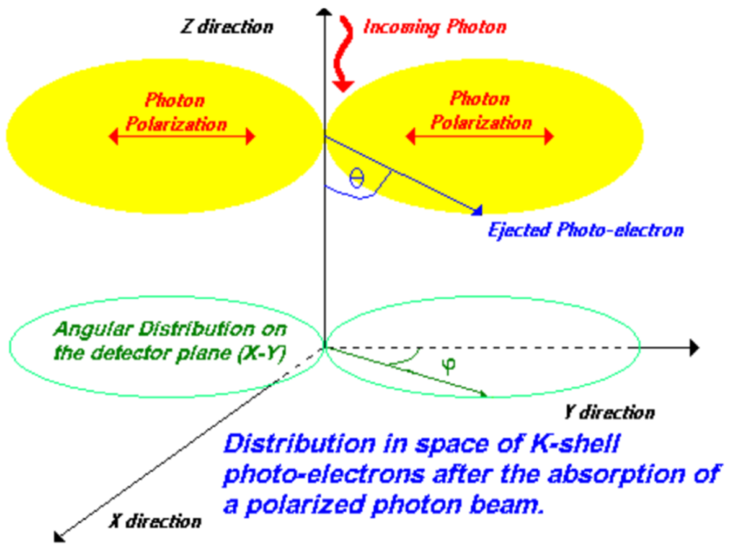
- Missions in the 70s used Thomson/Bragg polarimeters.
  - Bragg polarimetry is dispersive (1 angle at a time)
  - Thomson polarimetry is non-imaging and for E$\gamma$ > 5 keV.

- Jump to photoelectric (effect) polarimeters made (more sensitive to polarization).

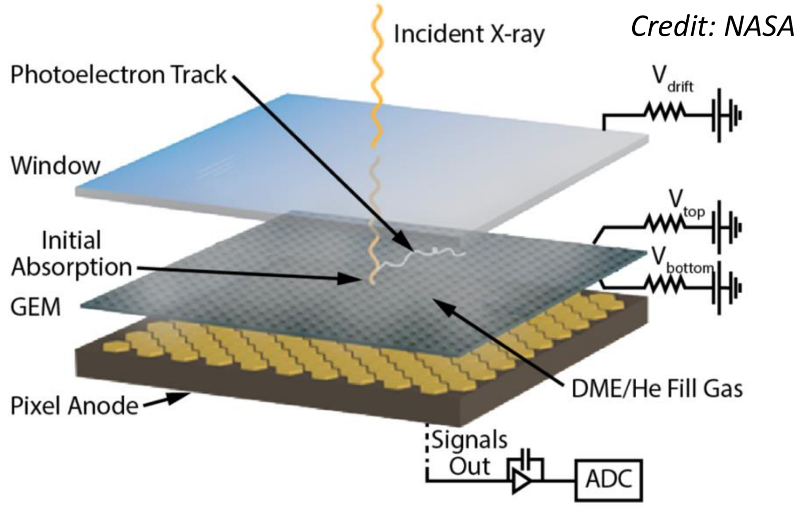
- Electron is ejected from an inner shell. Emission direction is non-uniform, but peaks around the E-field of the photon. Slowed by ionizing collisions and scattered by Coulomb diffusion of nuclei, leaves a string of election-ion pairs in the absorber.

- By measuring the angular distribution (modulation curve) of the ejected photo-electron "tracks" (and their direction), we can retrieve the original electron information and the polarization of the photon can be derived.

- Cross-section of s-electrons: $\frac{\partial\sigma}{\partial\Omega} = r_0^2 \frac{Z^5}{137^4}\left(\frac{mc^2}{h\nu}\right)^{7/2} \frac{4\sqrt{2sin^2\theta cos^2\varphi}}{(1-\beta\cos\theta)^4}$




Costa, E., Soffitta, P., Bellazzini, R. *et al.*. *Nature* **411,** 662–665 (2001). https://doi.org/10.1038/35079508

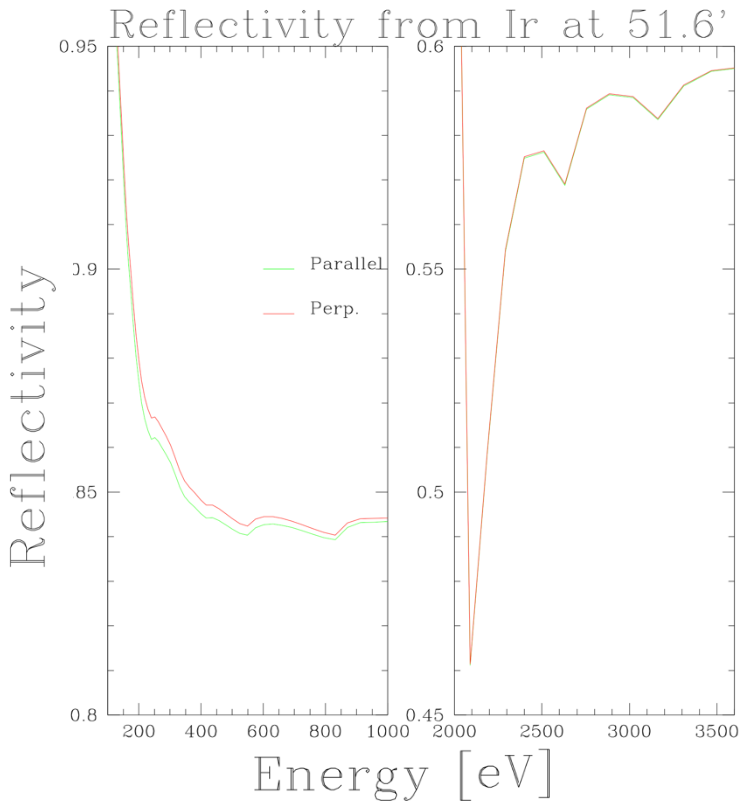



*Credit:* https://ixpe.msfc.nasa.gov/for_scientists/presentations/Muleri_IXPE_Catiadas.pdf
https://ixpe.msfc.nasa.gov/for_scientists/papers/2017spie_0829_sgro.pdf

# Fresnel Equations

- The amplitude coefficients of transmission and reflection for s and p polarization (Transverse-Electric [TE] and Transverse-Magnetic [TM]) around infinitely smooth interface of two materials, given by Fresnel equations.

- Use Maxwell's equations, keeping $E_{\parallel}$ and $H_{\parallel}$, $D_{\perp}$ and $B_{\perp}$ continuous across the interface.




**NB:**
**E** = electric field,
**D** = electric displacement field,
**B** = magnetic field,
**H** = magnetic field strength

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https://www.astro.umd.edu/~richard/ASTR680/schwartz_optics.pdf

# Fresnel Equations

- We need to consider polarization to apply the boundary conditions.
- The reflected wave amplitude of the parallel component:

$$r_p = \frac{\left|n^2 \cos\theta - \sqrt{n^2 - sin^2\theta}\right|^2}{\left|n^2 \cos\theta + \sqrt{n^2 - sin^2\theta}\right|^2}$$

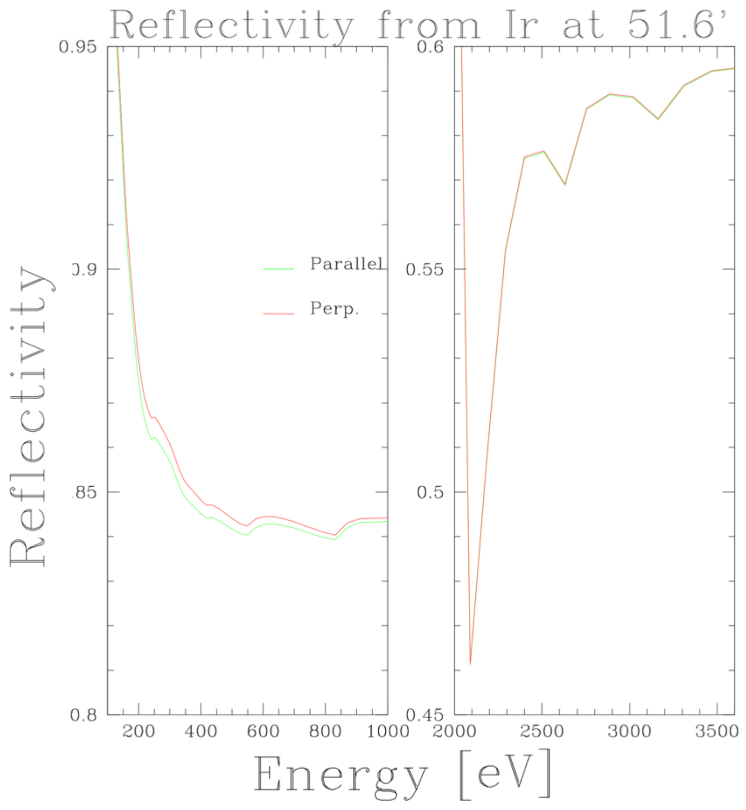
- The perpendicular component of the electric vector:

$$r_s = \frac{\left|\cos\theta - \sqrt{n^2 - sin^2\theta}\right|^2}{\left|\cos\theta + \sqrt{n^2 - sin^2\theta}\right|^2}$$

- The squared amplitudes of the complex numbers $r_p$ and $r_s$ are the reflectivities, such that for unpolarized X-rays the reflectivity becomes:

$$\frac{\left|r_p\right|^2 + |r_s|^2}{2}$$

- and $r_p \simeq r_s$ for X-rays, since $n \simeq 1$.

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